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Ideals in Quotient Semirings

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ABSTRACT

Since the class of quotient rings is contained in the class of quotient semirings, in this paper, we will make an intensive study of the properties of quotient semirings as compared to similar properties of quotient rings. The main aim of this paper is that of extending some well-known theorems in the theory of quotient rings to the theory quotient semirings.

Keywords: quotient semirings, weakly prime ideals, weakly primal ideal, semidomain like semirings

1. INTRODUCTION

Semirings are natural topic in algebra to study because they are the algebraic structure of the set of natural numbers. Semirings also appear naturally in many areas of mathematics. For example, semirings are useful in the area of theoretical computer science as well as in the solution of problem in graph theory and optimization. In structure, semirings lie between semigroups and rings. The class of rings is contained in the class of semirings [7,8]. Therefore, all of the properties given here apply to rings.

This paper generalizes some well-known result on quotient rings in commutative rings to commutative semirings. The maindifficulty is figuring out what additional hypotheses the semiring or ideal must satisfy to get similar results. Quotient semirings are determined by equivalence relations rather than by ideal as in the ring case. There are many different definitions of a quotient semiring appearing in the literature. P.J. Allen ([2]) introduced the notion of Q-ideal and a construction process was presented by which one can build the quotient structure of a semiring modulo a Q-ideal (also see the results listed in [3-5,7,8]). If I is an ideal of a semiring R, then Golan has presented the notion quotient semiring R/I, but this definition is different from the definition of Allen (see Section 2). Here we follow the definition of Golan. The main part of this paper is devoted to stating and proving analogues to several well-known theorems in the theory of quotient rings (see Section 2).

In order to make this paper easier to follow, we recall here various notions from semiring theory which will be used in the sequel. A commutative semiring R

is defined as an algebraic system $(R,+,.)$ such that $(R,+)$ and $(R,.)$ are commutative semigroups, connected by $a(b+c)=ab+ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r+0=r$ and $r0=0r=0$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative with non-zero identity. A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a+b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) J is an ideal such that if $x, x+y \in J$ then $y \in J$ (so $\{0\}$ is a k-ideal of R). A prim ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. A primary ideal P of R is a proper ideal of R such that, if $xy \in P$ and $x \notin P$, then $y \in \sqrt{P} = \{r \in R : r^n \in P$ for some positive integer $n\}$. A semiring R is said to be a semidomain if $ab=0$ $(a, b \in R)$, then either $a=0$ or $b=0$. A semifield is a semiring in which non-zero elements from a group under multiplication. An element a of a semiring R is called zero-divisor of R if there exists $0 \neq b \in R$ such that $ab=0$ (not here that we include 0 in the set of zero-divisors of semiring). The collection of all zero-divisors of R will be denoted by $Z(R)$. Furthermore, the subset $\{a \in R$: there exists a positive integer such that $a^n=0\}$ of $Z(R)$ consisting of the nilpotent elements of R will be denoted by nil (R), the nilradical of R.

2. RESULTS

Quotients semirings are determined by equivalence relations rather than by ideals as in the ring case. If I is an ideal of semiring R, we define a relation on R, given by $r_1 \sim r_2$ if and only if there exist $a_1, a_2 \in I$ satisfying $r_1 + a_1 = r_2 + a_2$. Then is an equivalence relation on R, and we denote the equivalence class of r by $r+I$ and these collection of all equivalence classes by R/I. Golan shows that R/I is a semiring with $(r+I)+(s+I)=r+s+I$ and $(r+I)+(s+I)=rs+I$ [7]. Our starting point is the following lemma.

Lemma 2.1 *Let I be an ideal of a semiring R. Then the following hold:*

(1) *If $a \in I$, then $a+I=I$.*

(2) *If I is a k-ideal of R and $a \in I$, then $a+I=b+I$ for every $b \in R$ if and only if $b \in I$. In particular, $c+I=I$ if and only if $c \in I$.*

Proof. (1) Since $a+0=0+a$, we conclude that $a \sim 0$; hence $a+I=0+I$.

(2) Let $a+I=b+I$ for every $b \in R$. Then $a+u=b+v$ for some $u, v \in I$; so $b \in I$ since I is a k-ideal. The other implication follows from (1) and the fact I is a k-ideal of R. □

Lemma 2.2 *Let I and J be ideals of a semiring R with $I \subseteq J$. Then the following hold:*

(1) *$J/I=\{a+I: a \in J\}$ is an ideal of R/I. In particuar, if J is a k-ideal of R, then J/I is a k-ideal of R/I.*

(2) *If $1+I \in J/I$, then $R/I=J/I$.*

(3) *If $a+I$ is a invertible element of R/I with $a+I \in J/I$, then $R/I=J/I$.*

Proof. (1)Cleary, $0+I \in J/I$. Let $a+I$, $b+I \in J/I$ and $r+I \in R)/I$. It is easy to see that $(a+I)+(b+I)=a+b+I \in J/I$ and $(r+I)(a+I)=ra+I \in J/I$. Thus J/I is an ideal of R/I. Finally, assume that $u+I \in J/I$ and $(u+I)+(v+I)=u+v+I \in J/I$, where $u \in J$ and $v \in R$. It then follows that $u+v+t_1=c+t_2$ for some $t_1, t_2 \in I$ and $c \in J$; hence $v \in J$ since J is a k-ideal. Thus $v+I \in J/I$, and the proof is complete.

(2) Let $x+I \in R/I$. Then $(x+I)(1+I)=x+I \in J/I$; hence $R/I \subseteq J/I$, as required.

(3) Follows from (2). □

Theorem 2.3 *Let I be an ideal of a semiring R. Then the following hold:*

(1) *If L is an ideal of R/I, then $L = J/I$ for some ideal J of R.*

(2) *If P is a k-ideal of R with $I \subseteq P$, then P is a prime ideal of R if and only if P/I is a prime ideal of R/I.*

(3) *I is a prime k-ideal of R if and only if R/I is a semidomain. In particular, (0) is prime if and onlyif only R is a semidomain.*

Proof. (1) Assume that $J = \{r \in R : r + I \in L\}$ and let $a \in I$. Then by Lemma 2.1, $a + I = 0 + I \in L$; hence $I \subseteq J$. Let $a,b \in J$ and $r \in R$. Then $(a + I)+(b + I) = a + b + I \in L$; so $a + b \in J$. Similarly, $ra \in J$. Thus J is an ideal of R. Finally, it is easy to see that $L = J/I$.

(2) Let P be a prime ideal of R. Suppose that $r + I$, $s + I \in P/I$ are such that $(r + I)(s + I) = rs + I \in P/I$, where $r, s \in R$. Then $rs + I = P + I$ for some $p \in P$. This implies that $rs \in P$ since P is a k-ideal. Then P prime gives either $r \in P$ or $s \in P$; so either $r + I \in P/I$ or $s + I \in P/I$ by Lemma 2.2. Conversely, suppose that P/I is prime. Let a, $b \in R$ such that $ab \in P$. Then by Lemma 2.1, $(a + I)(b + I) = ab + I = 0 + I \in P/I$; thus either $a + I \in P/I$ (so $a \in P$) or $b + I \in P/I$ (so $b \in P$), as required.

(3) Let I be a prime ideal of R and let $a + I$ and $b + I$ be elements of R/I such that $(a + I)(b + I) = ab + I = 0 + I$, where a, $b \in R$. By Lemma 2.1 (2), $ab \in I$; so either $a \in I$ or $b \in I$. Therefore, by Lemma 2.1 (1), either $a + I = I$ or $b + I = I$. Thus $a + R/I$ is semidomain. The proof of the other implication is similar. □

A proper ideal I in a semiring R is said to be maximal (resp. k-maximal)if J is an ideal (resp. a k-ideal) in R such that $I \subseteq J$, then $I = R$. Moreover, it is clear that if I, J and L are ideals of R with $I \subseteq J$, $I \subseteq L$ and $J/I = L/I$, then $J = L$.

Theorem 2.4 Let P be a proper k-ideal of a semiring R. Then the following hold:

(1) P is a maximal k-ideal of R if and only if R/I is a semifield.

(2) If I is an ideal of R with $I \subseteq P$, then P is a maximal k-ideal of R if and only if P/I is a maximal ideal of R/I.

Proof. (1) Let P be a maximal ideal of R. It suffices to show that every non-zero element $a + P$ of R/P is invertible. By Lemma 2.1, $a \notin P$; hence $P + Ra = R$ by maximality of P. There exist $r \in R$ and $p \in P$ such that $ra + p = 1$. It then follows from Lemma 2.1 that $(r + P)(a + P) = 1 + P$. Thus $a + P$ is invertible. Conversely, assume that R/P is a semifield and $P \subsetneq J$ for some k-ideal J of R; we show that $J = R$. Then there is an element $b \in J\text{-}P$ such that $b + P$ is invertible in R/P, so $(b+P)(c+P) = bc + P = 1+P$ forsome $c + P \in R/P$. Since J is a k-ideal, we conclude that $1 \in J$, as needed.

(2) Suppose that P is a maximal k-ideal of R and let L be a k-ideal of R/I such that $P/I \subsetneq U$. There exists a k-ideal J of R such that $P/I \subsetneq L = J/I$ by Theorem 2.3 (1), so $P \subsetneq J$; hence $J = R$. Thus $L = R/I$. The other implication is similar.

If R is a semiring, then R is Noetherian (resp. Artinian) if any non-empty set of k-ideals of R has a maximal member (resp. minimal member) with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending condition) on k-ideals. It is easy to see that If I and J are k-ideals of R, $I + J$ is a k-ideal of R, and an intersection of a family of k-ideals of R is k-ideal.

Theorem 2.5 Let I be a k-ideal of a semiring R. R is Noetherian (resp. Artinian) if and only if both I and R/I are Noetherian (resp. Artinian).

Proof. Let $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq J_{n+1} \subseteq \cdots$ be an ascending chain of k-ideals of R. Then $J_1 \cap I \subseteq J_2 \cap I \subseteq \cdots \subseteq J_n \cap I \subseteq J_{n+1} \cap I \subseteq \cdots$ is an ascending chain of k-ideals of I, and so there is a positive integer s such that $J_s \cap I = J_{s+i} \cap I$ for all positive integer i. So $(J_1 + I)/I \subseteq (J_2 + I)/I \subseteq \cdots \subseteq (J_n + I)/I \subseteq (J_{n+1} + I)/I \subseteq \ldots$ is a chain of k-ideals of R/I. Thus there exists a positive integer t such that $(J_t + I)/I = (J_{t+i} + I)/I$ for all positive integer i, so that $I + J_t = I + J_{t+i}$ for all i. Let $u = \max\{s,t\}$. We show that, for each positive integer i i, $J_u = J_{u+i}$. Since the inclusion $J_u \subseteq J_{u+i}$ is trivial, we will prove the reverse inclusion. Let $x \in J_{u+i}$. Since $x \in I + J_{u+i} = I + J_u$, we must have $x=a+b$ for some $a \in I$ and $b \in J_u \subseteq J_{u+i}$. Hence $a \in J_{u+i}$, since it is a k-ideal. It follows that $a \in I \cap J_{u+i} = I \cap J_u$; hence both a and b belong to J_u and $x \in J_u$. Thus R is Noetherian. Convesely, assume that R is Noetherian. By Theorem 2.3, is an ascending chain of k-ideals of R/I must have the form $J_1/I \subseteq J_2/I \subseteq \cdots \subseteq J_n/I \subseteq J_{n+1}/I \subseteq \cdots$, where $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq J_{n+1} \subseteq \cdots$ is an ascending chain of k-ideals of R all of which contain I. Since the latter chain must eventually become stationary, so must the former. Thus R/I is Noetherian. Since every subideal of I is an ideal of R, it is clear from the definition of Noetherian semiring that I is Noetherian. The Artinian case can be proved in a very similar manner tothe way in which was proved above, and we omit it.

An ideal I of a semiring R is strongly irreducible if for ideals J and K of J, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$.

Lemma 2.6 *Let I be an ideal of a semiring R. If J,K and L are k-ideals of R containing I, then $(J/I) \cap (K/I)=L/I$ if and only if $J \cap K$.*

Proof. Suppose that $(J/I) \cap (K/I) = L/I$; we show that $J \cap K = L$. Let $x \in J \cap K$. Then $x + I \in (J/I) \cap (K/I) = L/I$ so $x + a_1 = y + a_2$ for some $y \in L$ and $a_1, a_2 \in I$; thus $x \in L$ since L is a k-ideal. Thus $J \cap K \subseteq L$. For the reverse inclusion, suppose that $z \in L$. Then $z + I \in (J/I) \cap (K/I)$; so $z + b_1 = u + b_2$ and $z + c_1 = v + c_2$ for some $b_1, b_2, c_1, c_2 \in I$, $u \in k$ and $v \in J$. Thus $z \in J \cap K$ since K and J are k-ideals, and so we have equality. The other implication is similar. □

Theorem 2.7 *Let R be a semiring, I an ideal of R and J a strongly irreducible k-ideal of R with $I \subseteq J$. Then J/I is a strongly irreducible ideal of R/I.*

Proof. Let N and M be k-ideals of R/I such that $N \cap M \subseteq J/I$. Then there are k-ideals K,H of R such that $N = K/I$ and $M = H/I$ by Theorem 2.3; hence Lemma 2.7 gives $K \cap H \subseteq J$. Since J is strongly irreducible it follows that either $K \subseteq J$ or $H \subseteq J$; hence either $N = K/I \subseteq J/I$ or $M = H/I \subseteq J/I$. So J/I is strongly irreducible. □

A semiring R is called semidomainlike semiring, if $Z(R) \subseteq$ nil (R). A classic result of commutative semiring theory is that an ideal P is prime if and only if R/P is a semidomain (see Theorem 2.3 (3)). The following theorem is a parallel result for semidomainlike semirings.

Theorem 2.8 *Let P be a proper k-ideal of a semiring R. Then P is primary if and only if R/P is a semidomainlike semiring.*

Proof. Assume that P is a primary ideal of R and let $a + P \in Z(R/P)$. Then there exists a non-zero element $b + P$ of R/P such that $(a + P)(b + P) = 0 + P$; so $ab \in P$ by Lemma 2.1. If $b \in P$, then $b + P = 0 + P$ (see Lemma 2.1),which is a contradiction. Then P primary gives $(a + P)^n = a^n + P = 0 + P$ for somen by Lemma 2.1. Thus $a + P \in$

nil R/P and so R/P is a semidomainlike semiring. Conversely, let $ab \in P$, where $a, b \in R$. Then $(a + P)(b + P) = ab + P = 0 + P$ by Lemma 2.1. If $a + P = 0 + P$, then $a \in P$. Similarly, for $b + P = 0 + P$. So we may assume that $a + P \neq 0 + P$ and $b + P \neq 0 + P$. Therefore, by assumption $(a + P)^n = a^n + P = 0 + P$ for some n; hence $a^n \in P$. Thus P is primary. □

In general, a semidomainlike semirings is not necessarily a semidomain, but as the next result shows for semirings of the form $R/\sqrt{I}$, where I is anideal of R, the two concepts are equivalent.

Theorem 2.9 *Let I be an ideal of a semiring R. Then $R/\sqrt{I}$ is semidomainlike if and only if $R/\sqrt{I}$ is a semidomain. In particular, $R/nil(R)$ is semidomailike if and only if $R/nil(R)$ is a semidomain.*

Proof. Let $R/\sqrt{I}$ be semidomainlike, and let $(a + \sqrt{I})(b + \sqrt{I}) = ab + \sqrt{I} = 0 + \sqrt{I}$ in $R/\sqrt{I}$ with $a + \sqrt{I} \neq 0 + \sqrt{I}$. Then $ab \in \sqrt{I}$ by Lemma 2.1 and $a \notin \sqrt{I}$. Since $R/\sqrt{I}$ is semidomainlike, $\sqrt{I}$ is primary by Theorem 2.7. Therefore, $b^m \in \sqrt{I}$ for some positive integer m, whence $b \in \sqrt{I}$, $b + I = 0 + I$, and $R/\sqrt{I}$ is a semidomain. The other implication is similar. □

Let R be a semiring. We define a proper ideal I of R to be weakly primary (resp. weakly prime) if $0 \neq ab \in I$ implies $a \in I$ or $b^m \in I$ for some positive integerm (resp. $a \in I$ or $b \in I$) [1, 6]. So a primary ideal (resp. prime ideal) is a weakly primary (resp. weakly prime). However, since 0 is always weakly primary (resp. weakly prime) by definition, a weakly primary ideal (a weakly prime ideal) need not be primary (resp. prime). Clearly, every weakly prime is weakly primary.

Theorem 2.10 *Let R be a semiring, I an ideal of R and P a k-ideal of R with $I \subseteq P$. Then the following hold:*

(1) If P is a weakly primary ideal (resp. weakly prime ideal) of R, then P/I is aweakly primary ideal (resp. weakly prime ideal) of R/I.

(2) If both I and P/I are weakly primary (resp. weakly prime ideal) ideal, then P is weakly primary ideal (resp. weakly prime ideal).

Proof. (1) Assume that P is weakly prime and let $a + I$ and $b + I$ be elements of R/I such that $(0 + I) \neq (a + I)(b + I) = ab + I \in P/I$, so $0 \neq ab \in P$ by Lemma 2.1. Then P weakly primary gives either $a \in P$ or $b^n \in P$ for somen. If $a \in P$, then $a + I \in P/I$ by Lemma 2.2. So suppose that $b^n \in P$. It follows that $(b + I)^n = b^n + I \in P/I$. Thus P/I is weakly primary.

(2) Let $0 \neq ab \in P$, where $a, b \in R$. If $ab \in I$, then I weakly primary gives either $a \in I \subseteq P$ or $b^s \in I \subseteq P$ for some s. So we may suppose that $ab \in I$. By Lemma 2.1, we must have $(0 + I) \neq (ab + I) = (a + I)(b + I) \in P/I$; so *either* $a + I \in P/I$ or $(b + I)^m = b^m + I \in P/I$ for some m since P/I is weakly primary. If $a + I \in P/I$, then $a \in P$ by Lemma 2.1. If $b^m + I \in P/I$, then $b^m \in P$. Thus P is weakly primary. □

Let I be an ideal of a semiring R. An element $a \in R$ is called weakly prime to I if $0 \neq ra \in (r \in R)$ implies that $r \in I$, and let $p(I)$ be the set of elements of R that are not weakly prime to I. 0 is always weakly prime to I. A proper ideal I of R is called weakly primal if the set $P = p(I) \cup \{0\}$ form an ideal: this ideal is called the weakly adjoint ideal P of I. Let R be a commutative semiring which is not a semidomain. Then 0 is a 0-weakly primal ideal of R (by definition). Let I be an ideal of R and A a subset of R. We say that A satisfies (*) if A

is exactly the set of elements of R that are not weakly prime to I. We use the notation A^* to refer to the non-zeroelements of A.

Theorem 2.11 *Let J be a weakly prime k-ideal of a semiring R and I a proper k-ideal of R with $J \subseteq I$. Then I is a weakly primal ideal of R if and only if I/J is a weakly primal ideal of R/J. In particular, there is a bijective correspondence between the weakly primal ideals of R containing J and the weakly primal ideals of R/J.*

Proof. First suppose that I is a P-weakly primal ideal of R with $J \subseteq I$. Then by [3, Remark 3.2 and Theorem 3.4], $J \subseteq P$ and P is a weakly prime ideal of R; hence P/J is a weakly prime ideal of R/J by Theorem 2.10. It suffices to show that $(P/J)^*$ satisfies (*). Let $a + J \in (P/J)^*$, where $a \in P$. Since by Lemma 2.1, $0 + J \neq a + J$, we must have $a \neq 0$ and a is not weakly prime to I; hence there exists $r \in R - I$ such that $0 \neq ra \in I$. If $0 \neq ra \in J$, then J weakly prime gives $r \in J$, which is a contradiction since $r \notin I$. So we may assume that $0 \neq ra \notin J$. It follows that $0 \neq (r + J)(a + J) \in I/J$ with $r + J \notin I/J$, so $a + J$ is not weakly prime to I/J. Now assume that $b + J \neq 0 + J$ is not weakly prime to I/J, where $b \in I$. Then there exists $c + J \in R/J - I/J$ such that $0 \neq (c + J)(b + J) = cb + J \in I/J$, so $cb \in I$ with $c \notin I$ by Lemma 2.1. Thus $b \neq 0$ is not weakly prime to I. Therefore, $b + J \in (P/J)^*$ and the proof is complete.

Conversely, suppose that I/J is a P/J-weakly primal ideal of R/J; we show that I is a P-weakly primal ideal of R. By [3, Theorem 3.4] and Theorem 2.10, P is a weakly prime ideal of R. It is enough to show that P^* satisfies (*). Let $a \in P^*$. By [3, Remark 3.2], we can assume that $a \notin J$. As J is a weakly prime ideal and $0 \neq a + J \in P/J$, there exists $r + J \in R/J–I/J$ such that that $0 \neq (a + J)(r + J) = ar + J \in I/J$; hence $0 \neq ar \in I$ with $r \notin J$. Thus a is not weakly prime to I. Now assume that a is not weakly prime to I (so $a \neq 0$); we show that $a \in P$. We can assume that $a \notin I$. Then there is an element $r \in R - I$ such that $0 \neq ar \in I$. Therefore, $0 \neq (r + J)(a + J) = ra + J \in I/J$ with $r + J \notin I/J$; hence $a + J \in (P/J)^*$ since I/J is P/J-weakly primal. Thus $a \in P$, as required. □

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