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Stochastic Orders in Risk-Based Overbooking

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ABSTRACT

In this article, we propose a risk-based overbooking model, in which an overbooking limit is chosen such that an expected profit is maximized. We derive an optimality condition, when a show demand given a total number of reservations follows a binomial distribution. We also determine a directional change of the optimal overbooking limit with respect to various model parameters, such as a show-up probability, a per-unit revenue and a per-unit oversale cost. Finally, we show that the optimal expected profit decreases, if the show-up probability increases.

Keywords: stochastic model applications, operations research, revenue management.

1. INTRODUCTION

Overbooking occurs when a total number of reservations is greater than available capacity. Companies practice overbooking to protect themselves against cancellations and no-shows. In the passenger airline industry, about 15% of the total seats on a typical flight would be unsold if overbooking is not considered [1]. In 2005, Lufthansa credited the practice of overbooking for a revenue increase of one hundred million Euros [2]. Besides passenger airlines, overbooking is also found in air freight, business hotels, and rental cars.

An *overbooking limit (authorization level)* is the maximum number of booking requests to be accepted. In a *risk-based* overbooking model, an optimal overbooking limit is determined such that an expected profit is maximized. The profit is the total revenue minus *oversale cost*, which occurs whenever show demand exceeds the capacity. The *show demand (show-up)* refers to a reservation that survives to the time of service. The overbooking limit is set before any bookings arrive; during the booking horizon, the company accepts bookings until total reservations exceed the authorization level. If it is set too high, the company faces a high risk of oversale. On the other hand, if it is set too low, the company incurs an opportunity cost, the expected revenue gain from an additional booking. Under a risk-based policy, an optimal overbooking limit is balancing the expected oversale cost and the potential additional revenue from more bookings.

In this article, we propose a risk-based overbooking model. Our research objectives are as follows. First, we derive a closed-form expression for the optimal overbooking limit. Second, we determine how the optimal overbooking limit changes, when we vary model parameters, such as a probability that a reservation shows up, a probability distribution of the total requests, a per-unit revenue and a per-unit oversale cost. We show that the optimal overbooking limit increases, when the show-up probability decreases or the ratio of the revenue to the oversale cost increases. It is not affected by the distribution of the total booking requests. Third, we determine the directional change of the optimal expected profit as the show-up probability is varied.

The literature on overbooking problems dates back to Beckmann [3]. An overview of the overbooking problem can be found in, e.g., Chapter 4 Talluri and van Ryzin [4] and Chapter 9 Phillips [5]. Section 4.2.1 in Talluri and van Ryzin [4] presents an analysis when the number of total requests always exceeds the overbooking limit. In other words, their study does not take into account the distribution of the total requests, whereas ours does. Beckmann [3] assumes that the show demand follows a general continuous distribution; consequently, the expected profit is continuously differentiable, and the optimality condition can be found using a theorem in calculus. However, our show demand follows a discrete distribution, our expected profit is not differentiable, and discrete optimization techniques have to be employed. (The show demand is a binomial random variable, if the number of total reservations is given.) Moreover, we obtain some comparative statics and/or sensitivity analysis, when model parameters are varied.

To obtain some comparisons of risk-based overbooking limits and optimal expected profits, we apply stochastic orders. To the best of our knowledge, stochastic comparison in overbooking has not been published. References on the topic can be found in, e.g., Müller and Stoyan [6] and Shaked and Shanthikumar [7]. Recent papers on its applications in operations research include, e.g., Gupta and Cooper [8] and Cooper and Gupta [9]. Gupta and Cooper [8] debunk the idea behind a yield-improvement project that stochastically larger yield rates are preferable. They show that a yield rate that is smaller in the convex order leads to larger expected profit. Like ours, Cooper and Gupta [9] apply stochastic comparison methods in revenue management (RM) problem. Their study is concerned with a booking control decision, whereas ours is concerned with an overbooking decision. The main objective of booking control is to find an optimal mix of demand, whereas that of overbooking is to increases capacity utilization. Both booking control and overbooking problems are important parts of RM, whose objective is "to maximize revenue by selling the right product to the right customer at the right time for the right price ([10] page 52)." Survey papers on RM are, e.g., Weatherford and Bodily [11], McGill and van Ryzin [12] and Chiang et al. [13].

The rest of the paper is organized as follows. We present a risk-based overbooking model in Section 2. The main result is given in Section 3. We conclude with some thoughts on future research directions in Section 4.

2. FORMULATION

Throughout this article, N denotes the set of natural numbers, Z_+ the set of nonnegative integers, R the set of real numbers, R^n the n-dimensional Euclidean space, and $[y]^+ = \max(y, 0)$ the positive part of $y \in R$.

Consider a company that uses a risk-based policy to manage a fixed capacity κ. It wants to determine an overbooking limit $x \in N$ that maximizes its expected profit- expected total revenue minus expected oversale cost. Let D be the total number of booking requests that it receives, which is assumed to be an N-valued random variable. The company will book up to x, and the number of accepted requests (reservations) is $\min(x, D)$.

The total revenue the company earns equals the per-unit revenue $p > 0$ times the number of accepted requests. Given that the number of accepted requests is n, let $Z(n)$ be the number of reservations that show up at the time of service. If the number of show ups is greater than the capacity, then the airline incurs an oversale cost, which equals the per-unit oversale cost $h > 0$ times the number of show-ups exceeding the capacity. The company's expected profit is defined as follows:

$$\pi(x) = E\big[p \min(x,D) - h[Z(\min(x, D)) - \kappa]^{+}\big] \qquad (1)$$

In practice, the company offers the same service many times. For instance, an airline may operate a 9:00 AM flight from Bangkok to Chiang Mai seven days a week for the entire year. We can assume that the company is risk neutral. Therefore, our objective of maximizing the expected profit is appropriate.

When the show demand exceeds the available capacity, RM companies deal with denied-service situations differently. For instance, the US passenger airline first seeks customers who are willing to take a later flight in return for compensation (e.g., a certificate for future travel). If not enough volunteers are found, then it once or twice increases the compensation level. If still not enough, it will choose which additional passengers to bump. Those involuntarily bumped usually receive meals and possibly lodging while waiting for substitute transportation. For the business hotel, a bumped guest is usually accommodated at a nearby property, which is often in the same chain. In the estimating the denied-service parameter h, all direct costs as well as the loss of goodwill must be taken into account.

Assume that each reservation shows up independently, and that the probability of showing up is identical among all reservations. Then, for each $n \in N$ the show demand given the number of reservations n, denoted $Z(n)$, has a binomial distribution with parameters n and θ, where $\theta \in (0,1)$ is the *show-up probability*. Thompson [14] finds that the binomial distribution adequately fits data collected from Tasman Empire Airways.

Since all random variables take on nonnegative integers, the company can restrict attention to the overbooking limit x that is a natural number. Define the overbooking problem as $max\{\pi(x): x \in N\}$.

3. ANALYSES

In Proposition 1, we identify a sufficient condition such that the expected profit $\pi(x)$ is concave on Z_+ and derive an optimality condition. For each $x \in N$, let $\delta(x) = \pi(x) - \pi(x-1)$ be the difference of the expected profits evaluated at two consecutive positive integers. Recall that the expected profit function $\pi(x)$ is concave on Z_+, if the difference $\delta(x)$ is nonincreasing on N. (See the definition in, e.g., [15].) In Proposition 2, we characterize how the optimal overbooking limit changes, when the model parameter is varied. In Proposition 3, we show that the optimal expected profit increases, when the show-up probability decreases *ceteris paribus.*

Proposition 1. Assume that $P(Z(x - 1) \geq \kappa)P(D \geq x)$ is nondecreasing in x. Then,

1. The expected profit $\pi(x)$ is concave on Z_+.

2. If $\theta < p/h$, then the expected profit $\pi(x)$ is nondecreasing on Z_+; otherwise, an optimal overbooking limit is given as

$$x^* = \operatorname{argmax}\left\{x \in \mathbb{N} : P(Z(x-1) \geq \kappa) \leq \frac{p}{h\theta}\right\} \tag{2}$$

Proof. Denote $A(x) = \min(x, D) - \min(x - 1, D)$ and $B(x) = [Z(\min(x, D)) - \kappa]^+ - [Z(\min(x - 1, D)) - \kappa]^+$ for each $x \in N$. Clearly,

$$\delta(x) = pE[A(x)] - hE[B(x)]. \tag{3}$$

Since D is an N-valued random variable, we have that

$$E[\min(x, D)] = \sum_{j=0}^{\infty} P(\min(x,D) > j) = \sum_{j=0}^{x-1} P(D > j)$$

After some cancellation of terms, we get

$$E[A(x)] = P(D > x - 1) = P(D \geq x) \tag{4}$$

Let $T(x) = [Z(x) - \kappa]^+ - [Z(x - 1) - \kappa]^+$. Then, $E[B(x)|D = d] = E[T(x)]$ if $d \geq x$ and zero otherwise.

Therefore,

$$E[B(x)] = E[E[B(x)|D]] = E[T(x)]\, P(D \geq x) \tag{5}$$

We will compute $E[T(x)]$ by conditioning on $Z(x - 1)$. Recall that $Z(x)$ is a binomial random variable with parameters x and θ for each $x \in N$. Let Y be a Bernoulli random variable with mean $E[Y] = \theta$. Assume that Y is independent of $Z(x - 1)$. Then, $Z(x)$ has the same distribution as $[Z(x - 1) + Y]$. From the construction of $T(x)$, we get $E[T(x)|Z(x - 1)=j]=E[Y]$ if $j \geq \kappa$ and zero otherwise. Therefore,

$$E[T(x)] = E[E[T(x)|Z(x - 1)]] = \theta P(Z(x - 1) \geq \kappa) \tag{6}$$

Substituting (6) into (5), we obtain

$$E[B(x)] = \theta P(Z(x - 1) \geq \kappa)P(D \geq x) \tag{7}$$

The expression for the difference is obtained after substituting (4) and (7) into (3).

$$\delta(x) = pP(D \geq x) - \theta h\, P(D \geq x)P(Z(x-1) \geq \kappa) \tag{8}$$

$$= P(D \geq x)[p - \theta h\, P(Z(x-1) \geq \kappa)] \tag{9}$$

Recall that $p, h, \theta \geq 0$. Note that $P(D \geq x)$ is nonnegative and nonincreasing in x. If the assumption holds, then it follows from (8) that $\delta(x)$ is nonincreasing in x; i.e., the expected profit function $\pi(x)$ is concave on Z_+.

If $p/h > \theta$, then the term inside the square bracket in (9) is always nonnegative. Thus, $\delta(x) \geq 0$ for all $x \in N$. The expected profit $\pi(x)$ is nondecreasing on Z_+.

If $p/h \leq \theta$, then an optimal overbooking limit is the largest natural number such that $\delta(x) \geq 0$. (A local optimum of a concave function is also a global optimum.) Since the first term in (9) is nonnegative, the condition that $\delta(x) \geq 0$ becomes (2). □

In Proposition 1, Condition $argmax_x\{h\,\theta P(Z(x-1) \geq \kappa) \leq p\}$ can be argued intuitively as follows. Suppose that we have accepted $(x - 1)$ bookings, and we want to know whether or not to increase an overbooking limit. The right-hand side is the marginal revenue from one more booking. The left-hand side is the expected marginal cost: We would incur a marginal oversale cost of h, if all $(x - 1)$ bookings exceed the capacity [i.e., $Z(x-1) \geq \kappa$], and the xth booking shows up. The expected marginal cost equals the per-unit cost h times the probabilities of these two events. We continue to accept more bookings as long as the marginal revenue is at least the marginal cost. Surprisingly, the distribution of the total number of requests D does not affect the optimal overbooking limit.

Proposition 2. Suppose that the revenue-to-penalty ratio p/h increases, or that the show-up probability θ decreases. Then, the optimal overbooking limit increases ceteris paribus.

Proof. It follows from pp. 61–63 Müller and Stoyan [6] that $Z(x-1) \leq_{st} Z(x)$. In particular, $P(Z(x-1) \geq \kappa) \leq P(Z(x) \geq \kappa)$ for each $x \in N$. Function $P(Z(x-1) \geq \kappa)$ is nondecreasing in x. The directional change in the optimal overbooking limit with respect to the ratio p/h is obvious from (2).

Let $Z_\theta(x)$ be the binomial random variable with parameters (x, θ) and $x^*(\theta)$ the corresponding optimal overbooking limit, if the show-up probability is θ. For any two show-up probabilities $0<\theta_1 \leq \theta_2<1$, $p/(h\theta_1) \geq p/(h\theta_2)$, and $P\left(Z_{\theta_1}(x-1) \geq \kappa\right) \leq P\left(Z_{\theta_2}(x-1) \geq \kappa\right)$. The latter follows from the fact that $Z_{\theta_1}(x) \leq_{st} Z_{\theta_2}(x)$ for each $x \in N$. Recall that $P(Z_\theta(x) \geq \kappa)$ is nondecreasing in x. Thus, $x^*(\theta_1) \geq x^*(\theta_2)$. □

Results in Proposition 2 make sense economically. An optimal overbooking limit decreases, as the show-up probability increases. If the company anticipates that a large proportion of reservations would show up, then it should be more conservative and set a low overbooking limit in order to avoid possibly high oversale cost. An optimal overbooking limit increases, as the revenue-to-penalty increases. If the revenue is high or the penalty cost is low, then the company should be more aggressive and set a high overbooking limit.

Proposition 3. If the show-up probability increases, then the optimal expected profit decreases ceteris paribus.

Proof. When the show-up probability is $\theta \in (0, 1)$, let $\pi(x|\theta)$ be the expected profit function for each $x \in \mathrm{N}$, and $x^*(\theta)$ the optimal overbooking limit. Denote the corresponding optimal expected profit by $\pi^*(\theta) = \pi(x^*(\theta)|\theta)$. Assume $\theta_1 \leq \theta_2$. We want to show that $\pi^*(\theta_1) \geq \pi^*(\theta_2)$.

Let $Y_1(\theta), Y_2(\theta), \ldots$ be independent and identically distributed Bernoulli random variables with the common mean θ. Then, $Y_i(\theta_1) \leq_{\mathrm{st}} Y_i(\theta_2)$ for each $i \in \mathrm{N}$. Suppose that the number of total requests is $D = d$. It follows from Theorem 3.3.8 page 93 Müller and Stoyan [6] that

$$\left(Y_1(\theta_1), \ldots, Y_{\min(x,d)}(\theta_1)\right) \leq_{\mathrm{st}} \left(Y_1(\theta_2), \ldots, Y_{\min(x,d)}(\theta_2)\right).$$

Since $\left[\sum_{i=1}^{\min(x,d)} y_i - \kappa\right]^+$ is a bounded increasing function in $\left(y_1, \ldots, y_{\min(x,d)}\right)$, we have that

$$E\Big[\big[\sum_{i=1}^{\min(x,d)} Y_i(\theta_1) - \kappa\big]^+\big|D = d\Big] \leq E\Big[\big[\sum_{i=1}^{\min(x,d)} Y_i(\theta_2) - \kappa\big]^+\big|D = d\Big],$$

or equivalently

$$E\Big[\big[Z_{\theta_1}(\min(x,d)) - \kappa\big]^+\big|D = d\Big] \leq E\Big[\big[Z_{\theta_2}(\min(x,d)) - \kappa\big]^+\big|D = d\Big]$$

where $Z_\theta(min(x, d))$ is a binomial random variable with parameters $min(x, d)$ and θ. Using the conditional expectation formula and substituting the result into the expression of the expected profit, we get

$$\pi(x|\theta_1) \geq \pi(x|\theta_2) \tag{10}$$

for each $x \in \mathrm{N}$. (Recall that $h \geq 0$.) Thus,

$$\pi^*(\theta_1) = \pi(x^*(\theta_1)|\theta_1) \geq \pi(x^*(\theta_2)|\theta_1) \geq \pi(x^*(\theta_2)|\theta_2) = \pi^*(\theta_2)$$

The first inequality follows from the optimality of $x^*(\theta_1)$, and the second inequality follows from (10). ☐

Recall that the expected profit is the expected revenue less the expected oversale cost. The expected revenue $E[p\,min(x, D)]$ does not depend on the show-up probability. In other words, the show-up probability affects only the expected oversale cost $E[h[Z(min(x, D)) - \kappa]^+]$. Suppose that it is more likely for a reservation to survive to the time of service. Then, the company would expect to pay a larger denied-service cost and earn a smaller profit. In short, the optimal expected profit is decreasing in the show-up probability.

Note that the result in Proposition 3 holds in a more general setting. For instance, an oversale cost needs not to be linear in the number of customers that are denied service. Specifically, let $\phi_h(s)$ denote the oversale cost if the number of denied-service customers is s, and $\phi_p(n)$ the revenue if the number of reservations is n. Define the expected profit function as $\tilde{\pi}(x) = E\left[\phi_p(\min(x, D)) - \phi_h\left([\min(x, D) - \kappa]^+\right)\right]$. Proposition 3 holds for an arbitrary function ϕ_p and a bounded nonnegative increasing function ϕ_h.

4. CONCLUDING REMARKS

We propose an overbooking model and derive an optimal overbooking limit that maximizes the expected profit. The optimal overbooking limit is affected by the ratio of the per-unit revenue to the per-unit penalty cost, and the show-up probability. It increases, when the ratio increases, or the show-up probability decreases. However, it does not depend on the distribution of the total number of booking requests. Similarly, the optimal expected profit increases, when the show-up probability decreases.

There are several extensions to this article. For instance, the assumption that each reservation requires a single seat may be relaxed, and group bookings are allowed. Also, we may relax the assumption that each reservation shows up with the same probability. If the capacity is sold to multiple fare classes with different cancellation and refund policies, then the show-up probabilities may differ among reservations. We hope to pursue these or related problems in the future.

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