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Contributed Paper

On Representable Modules

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ABSTRACT

Let R be a commutative ring with non-zero identity. Our objective is to investigate representable modules and to examine in particular when submodules of such modules are representable.

Keywords : primary, pure, representable, secondary.

1. INTRODUCTION

The theory of secondary representation is a sort of the theory of primary decomposition of a module over a non-trivial commutative ring R. In [1] the following question was investigated: When are submodules of representable R-modules representable? In that paper [1, Theorem 2.3], it is shown that this is the case, when R is Von Neumann regular. In this paper, various properties of submodules of a representable module are considered. For example, we prove that over a commutative ring R (over a Dedekind domain R), if N is a primary submodule (non-zero pure submodule) of a representable R-module M, then N is representable and $\mathrm{Att(M)} = \mathrm{Att(N)} \cup \mathrm{Att(M/N)}$ (see Theorem 2.2 and Theorem 2.6).

Before stating some results let us introduce some notations and terminologies. Throughout this note all rings are commutative with identity and all modules are unitary. A proper submodule N of a module M over a ring R is said to be a primary submodule if for each $r \in R$, the R-endomorphism of M/N produced by multiplication by r is either injective or nilpotent, so $\mathrm{nilrad(M/N)} = P$ is a prime ideal of R, and N is said to be P-primary submodule. So N is primary in M if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then either $m \in M$ or $r^k M \subseteq N$ for some k. We say that M is a primary module if the zero submodule of M is a primary submodule of M.

An R-module M is secondary if $0 \neq M$ and, for each $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies that $\mathrm{nilrad(M)} = P$ is a prime ideal of R, and M is said to be P-secondary. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. So whenever an R-module M has a secondary representation, then the set of attached primes of M, which is uniquely determind, is denoted by Att(M) (see [5]).

Let N be an R-submodule of M. Then N is pure in M if any finite system of equations over N which is solvable in M is also solvable in N. So if N is pure in M, then $IN = N \cap IM$ for each ideal I of R. A submodule N of an R-module is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$.

Clearly, every pure submodule is relatively divisible submodule. An important property of Dedekind domains is that N is pure in M

if and only if N is RD-submodule of M (see [3, Theorem 4.5]).

2. SECONDARY MODULES

Our starting point is the following lemma:

Lemma 2.1 Let R be a commutative ring, M an R-module, and N a P-secondary R-submodule of M. If K is a primary submodule of M, then $N \cap K$ is P-secondary.

Proof. Let $a \in R$. If $a \in P$ then $a^n(N \cap K) \subseteq a^n N = 0$ for some n. Assume that $a \notin P$ and $m \in N \cap K$. Since $aN = N$, there is an element $s \in N$ such that $m = as$. As K is primary, it follows that either $a^n M \subseteq K$ for some n or $s \in K$. If $a^n M \subseteq K$, then $N = a^n N \subseteq a^n M \subseteq K$, and hence $a(N \cap K) = N \cap K$. If $s \in K$, then $m = as \in a(N \cap K)$. This gives $N \cap K = a(N \cap K)$ and the proof is complete.

Theorem 2.2 Let R be a commutative ring, M be a representable R-module, and N be a P-primary R-submodule of M. Then the following hold:

(i) N is representable. In particular, $\mathrm{Att}(N) \subseteq \mathrm{Att}(M)$.

(ii) $\mathrm{Att}(M) = \mathrm{Att}(N) \cup \mathrm{Att}(M/N)$.

(iii) M/N is P-secondary.

Proof. (i) Let $M = \sum_{i=1}^{m} M_i$ be a minimal secondary representation of M with $\mathrm{Att}(M) = \{P_1, \ldots, P_m\}$, $m_i \in M_i$ and $a_i \in P_i$. Then $a_i^{n_i} m_i = 0$ for some n_i, and we have $(a_i^{n_i} + P)(m_i + N) = 0$. Thus, either $a_i^{n_i s_i} \in P$ for some s_i or $m_i \in N$, and hence either $P_i \subseteq P$ or $M_i \subseteq N (1 \leq i \leq m)$. Clearly, $M_i \not\subset N$ for some i. If $M_i \not\subset N$, then $P_i = P$ (otherwise, if $c \in P - P_i$, then $c^t M \subseteq N$ for some t, so $M_i = c^t M_i \subseteq c^t M \subseteq N$, a contradiction). Since $P_1, \ldots, P_m$ are m different prime ideals of R, without loss of generality, we can assume that $M_1 \not\subset N$ and $M_i \subseteq N$, so $P = P_1$ and $P_i \not\subset P (2 \leq i \leq m)$. Then $N = N \cap M = M_2 + \ldots + M_m + (N \cap M_1)$, so N is representable by Lemma 2.1. Now the last part is immediate from the first part.

(ii) This follows from (i) and [5, Theorem 4.1].

(iii) We have $M/N = (N+M_1)/N \cong M_1/(M_1 \cap N)$, as required.

Lemma 2.3 Let R be a commutative ring, M be an R-module, and K be a representable R-submodule of M. If N is a P-primary submodule of M, then $N \cap K$ is representable.

Proof. We can assume that $K \not\subset N$. Let $K = \sum_{i=1}^{m} K_i$ be a minimal secondary representation of K with $\mathrm{Att}(M) = \{P_1, \ldots, P_m\}$, $m_i \in K_i$ and $a_i \in P_i$. Then $a_i^{n_i} m_i = 0$ for some n_i, and we have $(a_i^{n_i} + P)(m_i + N \cap K) = 0$. Thus, $a_i^{n_i} m_i \in N \cap K \subseteq N$, so either $P_i \subseteq P$ or $K_i \subseteq N (1 \leq i \leq m)$. Now we can assume that $K_1 \not\subset N$ and $K_i \subseteq N$. So $P = P_1$ and $Pi \not\subset P (2 \leq i \leq m)$ (the proof is similar to that the Theorem 2.2). Then $N \cap K = K \cap (N \cap K) = K_2 + \ldots + K_m + (N \cap M_1)$, hence $N \cap K$ is representable by Lemma 2.1.

Proposition 2.4 Let M be a module over a commutative ring R, and let the submodule N of M possess a primary decomposition. If K is a representable submodule of M, then $N \cap K$ can be expressed as an intersection of finitely many representable submodules.

Proof. Let $N = N_1 \cap N_2 \cap \ldots \cap N_s$, where N_i is primary, be a normal decomposition. Then $N \cap K = (N_1 \cap K) \cap \ldots \cap (N_s \cap K)$. Now the assertion follows from Lemma 2.3.

Lemma (2.5) Let R be a commutative ring, and let N be a pure submodule of R-module M. Then

(i) IN is an RD-submodule of IM for every ideal I of R.

(ii) aN is an RD-submodule of aM for every element a of R.

Proof. (i) Assume that $a \in R$ and let $J = Ra$. Then $IN \cap JIM = (N \cap IM) \cap IJM = N \cap IJM = IJN$ since N is pure in M. It follows that

$a(IN) = IN \cap a(IM)$, as required.

(ii) This follows from (i).

Theorem 2.6 Let R be a Dedekind domain, M be a representable R-module, and N be a non-zero pure submodule of M. Then

(i) N is representable. In particular, $\mathrm{Att}(N) \subseteq \mathrm{Att}(M)$.

(ii) $\mathrm{Att}(M) = \mathrm{Att}(N) \cup \mathrm{Att}(M/N)$.

Proof. (i) Let $M=\sum_{i=1}^{n} M_i$ be a minimal secondary representation of M with $\mathrm{Att}(M) = \{P_1, P_2, \dots, P_n\}$. Then there are elements $a_1, \dots, a_n$ of R such that $M=\sum_{i=1}^{n} a_i M$ where $a_i \notin P_i$, $a_i M = M_i$ $(1 \leq i \leq n)$, and $a_i \in \bigcap_{j=1, i\neq j}^{n} (0:M_j)$ (see [1, Theorem 2.3]).

Let $0 \neq a \in N$. Then $a=\sum_{i=1}^{n} a_i m_i$ for some $m_1, \dots, m_n \in M$. By the purity of N, there exist $b_1, \dots, b_n \in N$ such that $a=\sum_{i=1}^{n} a_i b_i$. Since $0 \neq a$, $a_i b_i \neq 0$ for some i, we have $a_i N \neq 0$. Assume that $a_{i_1} N \neq 0, \dots, a_{i_k} N \neq 0$, where $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. Clearly, $N=\sum_{j=1}^{k} a_{i_j} N$. By Lemma 2.5, for each $j (1 \leq j \leq k)$, $a_{i_j} N$ is pure in the P_{i_j}-secondary module M_{i_j}, so it is P_{i_j}-secondary by [2, Lemma 2.1], and the proof is complete. Now the last part is followed from the first part.

(ii) This follows from (i) and [5, Theorem 4.1].

Proposition 2.7 Let R be a Dedekind domain, M be an R-module, and N be a non-zero pure representable submodule of M. If M/N is P-secondary, then M and N are P-secondary.

Proof. Let $P' \in \mathrm{Att}(N)$. By [5, Theorem 2.2], there exists a quotient N/T of N such that N/T is P'-secondary with $\mathrm{nilrad}(N/T) = P'$. Set $\mathrm{nilrad}(M/T)=Q$. Clearly, $Q \subseteq P'$. If $a \in P'$, then $a^n N \subseteq T$ for some n. As N/T is pure in M/T (see [4, Consequences 18-2.2]) we get $a^n(N/T) = N/T \cap a^n(M/T)$, so $0 = N/T \cap a^n(M/T)$, hence $a^n M \subseteq T$. Therefore, $P' \subseteq Q$, so $Q=P'$. Since $(M/T)/(N/T) \cong M/N$, it follows that $P' \subseteq P$, and hence $P = P'$ since over R, every prime is maximal. Thus N is P-secondary. Now the assertion follows from [1, Lemma 2.1].

Theorem 2.8 Let R be a Dedekind domain, M be an R-module, and N be a non-zero pure submodule of M. Then M is representable if and only if M/N and N are representable.

Proof. If M is representable, then N and M/N are representable by Theorem 2.6 and [5, Theorem 2.4], respectively. Conversely, assume that M/N and N are representable and let $M/N=\sum_{i=1}^{n} M_i/N$ be a minimal secondary representation of M/N. Then by Proposition 2.7, $M=\sum_{i=1}^{n} M_i$ is secondary, as required.

The following results for RD-submodules of a primary module hold:

Proposition 2.9 Let R be a commutative ring, M be a primary R-module, and N be a RD-submodule of M. Then N is a primary submodule of M.

Proof. It is enough to show that M/N is a primary R-module. Assume that $r(a+N) = ra+N = 0$ for some $a \in M$ and $r \in R$. Then $ra=b$ for some $b \in N$, so there is an element $c \in N$ such that $rc=b$ by relatively divisiblity of N, and hence $r(a-c)=0$. Thus, either $a=c$ or $r^m M=0$ for some m since M is primary. Therefore, either $a+N=0$ or $r^m(M/N)=0$, and the proof is complete.

Theorem 2.10 Let R be a commutative ring and let N be an RD-submodule of R-module M. Then M is P-primary if and only if N and M/N are P-primary.

Proof. Assume that M is P-primary. Then M/N and N are P-primary by Proposition 2.9 and by the definition, respectively. Conversely, assume that N and M/N are P-primary and let $r \in R$. If $r \in P$, then $r^m(M/N) = 0$ and $r^m N=0$ for some m, hence $r^m M \subseteq N$ and $0 = r^m N = N \cap r^m M = r^m M$. Therefore, the R-

endomorphism of M produced by multiplication by r is nilpotent. Assume that $r \notin P$. If $sa=0$ ($s \in R$, $a \in M$), then $s(a+N) = 0$, so $a \in N$ since M/N is P-primary. As N is P-primary and $ra \in N$, we have $a=0$, as required.

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