



Almost Primal Ideals in Commutative Rings

Ahmad Y. Darani

Department of Pure Mathematics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran.

Author for correspondence; e-mail: yousefian@uma.ac.ir and youseffian@gmail.com

Received 4 August 2009

Accepted 16 August 2010

ABSTRACT

Let I be a proper ideal of a commutative ring R . An element $a \in R$ is called almost prime to I provided that $ra \in I - I^2$ (with $r \in R$) implies that $r \in I$. We denote by $A(I)$ the set of all elements of R that are not almost prime to I . I is called an almost primal ideal of R if the set $A(I) \cup I^2$ forms an ideal of R . In this paper we first provide some results on almost primal ideals. We also study the relations among the primal ideals, weakly primal ideals and almost primal ideals of R .

Keywords: almost prime ideal, primal ideal, weakly primal ideal, weakly prime ideal.

Throughout, R will be a commutative ring with identity. (However, in most places the existence of an identity plays no role.) By a proper ideal I of R we mean an ideal I with $I \neq R$. The concept of primal ideals in a commutative ring R introduced and studied in [1] (see also [2, 3]). Let I be an ideal of R . An element $a \in R$ is called prime (resp. weakly prime) to I if $ra \in I$ (resp. $0 \neq ra \in I$) (where $r \in R$) implies that $r \in I$. Denote by $S(I)$ (resp. $W(I)$) the set of elements of R that are not prime (resp. are not weakly prime) to I . A proper ideal I of R is said to be primal if $S(I)$ forms an ideal (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint ideal P of I . In this case we also say that I is a P -primal ideal of R . Note that if $r \in R$ and $a \in S(I)$, then clearly $ra \in S(I)$. So what we require for I being primal is that if a and b are not prime to I , then their

difference is also not prime to aI . Also, a proper ideal I of R is called weakly primal if the set $P = W(I) \cup \{0\}$ forms an ideal; this ideal is always a weakly prime ideal by [4, Proposition 4]. In this case we also say that I is a P -weakly primal ideal. Of course, a weakly prime ideal P of R is a proper ideal P with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. If R is not an integral domain, then 0 is a 0 -weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

In their paper [5], Bhatwadekar and Sharma introduced the concept of almost prime (resp. n -almost prime) ideals, where a proper ideal I of R is said to be almost prime if $a, b \in R$ with $ab \in I - I^2$ (resp. $ab \in I - I^n$) implies $a \in I$ or $b \in I$. Here we define two new concept of ideals: almost primal and n -almost primal ideals.

Definition 1. Let I be an ideal of R , and consider an integer $n \geq 2$. An element $a \in R$ is called almost prime (resp. n -almost prime) to I if $ra \in I - I^2$ (resp. $ra \in I - I^n$ (with $r \in I$) implies that $r \in I$.

Example 2. Let $R = Z/24Z$ and consider the ideal $I = 6Z/24Z$ of R . Clearly $I^2 = 12Z/24Z$. It is straight forward to show that $5 + 24Z$ is almost prime to I .

Remarks 3. Let I be an ideal of R . Denote by $A(I)$ the set of all elements of R that are not almost prime to I , and by $A_n(I)$ the set of all elements of R that are not n -almost prime to I (so $A_2(I) = A(I)$). Then:

- (1) Every element of I^2 (resp. I^n) is almost prime (resp. n -almost prime) to I .
- (2) If an element $a \in R$ is prime to I , then it is n -almost prime to I for every $n \geq 2$, but not conversely.

Lemma 4. Let I be an ideal of R . If $P = A(I) \cup I^2$ (resp. $P = A(I) \cup I^n$) is an ideal of R , then it is an almost prime (resp. n -almost prime) ideal of R .

Proof. Let $a, b \in R$ be such that $ab \in P - P^2$. Then, $rab \in I - I^2$ for some $r \in R - I$. Assume that $a \notin P$. Then a is almost prime to I . So $rb \in I - I^2$, implies that b is not almost prime to I , that is $b \in P$.

Definition 5. Let I be an ideal of R .

- (1) I is called almost primal if the set $P = A(I) \cup I^2$ forms an ideal of R . This ideal P is an almost prime ideal of R , called the almost prime adjoint ideal of I . In this case we also say that I is a P -almost primal ideal.
- (2) I is called n -almost primal if the set $P = A(I) \cup I^n$ forms an ideal of R . This ideal P is an n -almost prime ideal of R , called the n -almost prime adjoint ideal of I . In this case

we also say that I is a P - n -almost primal ideal.

Note that a 2-almost primal ideal is just an almost primal ideal.

The following example shows that all ideals of R need not necessarily be almost primal.

Example 6. Let $R = Z/24Z$ and consider the ideal $I = 6Z/24Z$ of R . Clearly $I^2 = 12Z/24Z$. Then, $(2 + 24Z)(3 + 24Z) \in I - I^2$ with $2 + 24Z, 3 + 24Z \in R - I$. So $2 + 24Z$ and $3 + 24Z$ are not almost prime to I . But $(2 + 24Z) + (3 + 24Z) = 5 + 24Z$ is almost prime to I . This shows that $A(I) \cup I^2$ is not an ideal of R . Therefore I is not almost primal.

Here we give several characterizations of almost primal ideals.

Proposition 7. Let I and P be proper ideals of R . For every integer $n \geq 2$, the following statements are equivalent:

- (1) I is P - n -almost primal.
- (2) For every $x \notin P - I^n$, $(I :_R x) = I \cup (I :_R x)$; and for every $x \in P - I^n$, $(I :_R x) \supseteq I \cup (I^n :_R x)$.
- (3) for every $x \notin P - I^n$, $(I :_R x) = I$ or $(I :_R x) = (I^n :_R x)$; and for every $x \in P - I^n$, $(I :_R x) \supseteq I$ and $(I :_R x) \supseteq (I^n :_R x)$.

Proof. 1 \Rightarrow 2) Assume that I is P - n -almost primal. Then $P - I^n$ consists entirely of elements of R that are not n -almost prime to I . Let $x \notin P - I^n$. Then x is n -almost prime to I . Clearly $(I :_R x) \supseteq I \cup (I^n :_R x)$. For every $r \in (I :_R x)$, if $rx \in I^n$, then $r \in (I^n :_R x)$, and if $rx \notin I^n$, then x n -almost prime to I gives $r \in I$. Hence $r \in I \cup (I^n :_R x)$, that is $(I :_R x) \subseteq I \cup (I^n :_R x)$. Therefore $(I :_R x) = I \cup (I^n :_R x)$.

Now assume that $x \in P - I^n$. Then x is not n -almost prime to I . So there exists $r \in R - I$ such that $rx \in I - I^n$. Hence $r \in (I :_R x) - (I \cup (I^n :_R x))$.

2⇒3) Let $x \notin P - I^n$. Since $(I :_R x)$ is an ideal of R and $(I :_R x) = I \cup (I^n :_R x)$, either $I \subseteq (I^n :_R x)$ or $(I^n :_R x) \subseteq I$. So either $(I :_R x) = I$ or $(I :_R x) = (I^n :_R x)$. Moreover, for every $x \in P - I^n$, $(I :_R x) \supseteq I \cup (I^n :_R x)$. Hence $(I :_R x) \supseteq I$ and $(I :_R x) \supseteq (I^n :_R x)$.

3⇒1) By (3), $P - I^n$ consists exactly of all elements of R that are not n -almost prime to I . Hence I is $P - n$ -almost primal.

Theorem 8. Let $n > 2$. Let P be an n -almost prime ideal of R . If P is not prime, then $P^2 = P^n$.

Proof. Suppose that $P^2 \neq P^n$. We show that P is prime. Assume that $a, b \in R$ are such that $ab \in P$. We have two cases $ab \notin P^n$ and $ab \in P^n$. In the former case, P n -almost prime gives $a \in P$ or $b \in P$. So we can assume that the latter case holds. First suppose that $aP \not\subseteq P^n$. Then $aP_0 \notin P^n$ for some $P_0 \in P$. In this case $a(b + P_0) \in P - P^n$, and P n -almost prime gives $a \in P$ or $b + P_0 \in P$. Hence $a \in P$ or $b \in P$. Thus we may assume that $aP \subseteq P^n$.

Similarly we can assume that $aP \subseteq P^n$. There exist $p, q \in P$ with $pq \notin P^n$ since $P^2 \neq P^n$. In this case $(a+p)(b+q) \in P - P^n$. So P n -almost prime gives $a+p \in P$ or $b+q \in P$. Therefore $a \in P$ or $b \in P$. Hence P is prime.

Let I be a proper ideal of R and let $n > 2$. Then $I - I^2 \subseteq A(I)$ (resp. $I - I^n \subseteq A_n(I)$). Hence, if I is a P -almost primal (resp. $P - n$ -almost primal) ideal of R , then $I \subseteq P$. This fact is used in the proof of the following result.

Theorem 9. Let $n > 2$. Let I be a $P - n$ -almost primal ideal of the commutative ring R such that P is a prime ideal. If $I^2 \neq I^n$, then I is primal.

Proof. It is enough to show that $P = S(I)$.

For every $a \in P$, if $a \in I^n$, then $a \in S(I)$ since $I \subseteq S(I)$; and if $a \notin I^n$, then a is not n -almost prime to I ; so, by Remark 3, a is not prime to I , that is $P \subseteq S(I)$. For the reverse containment assume that $a \in S(I)$. There exists $r \in R - I$ with $ra \in I$. If $ra \notin I^n$, then a is not n -almost prime to I and so $a \in P$. Suppose that $ra \in I^n$. If $aI \not\subseteq I^n$, there exists $r_0 \in I$ such that $ar_0 \notin I^n$. Now $a(r+r_0) \in I - I^n$ with $r+r_0 \in R - I$ gives $a \in P$. So assume that $aI \subseteq I^n$. If $rI \not\subseteq I^n$, there exists $c \in I$ with $rc \notin I^n$. Hence from $(a+c)r \in I - I^n$ with $r \in R - I$ we get $a+c \in P$. Further, $c \notin I^n$, for otherwise, $rc \in I^n$, a contradiction. Since $I - I^n \subseteq P$, we have $c \in P$. Consequently, $a \in P$. Thus assume that $rI \subseteq I^n$. Since $I^2 \neq I^n$, there exist $a_0, b_0 \in I$ with $a_0 b_0 \notin I^n$. Now $(a+a_0)(r+b_0) \in I - I^n$ with $r+b_0 \in R - I$ implies that $a+a_0 \in P$. Hence $a \in P$ again by the previous argument. Therefore in any case a lies in P , that is $S(I) \subseteq P$, and hence $P = S(I)$. So I is P -primal.

Definition 10. Let I be a proper ideal of R . I is said to be almost primary if, for $a, b \in R$, $ab \in I - I^2$ implies $a \in I$ or $b \in \sqrt{I}$. If $\sqrt{I} = P$, then I is called P -almost primary.

Theorem 11. Every almost primary ideal of a commutative ring R is almost primal. In particular, every almost prime ideal is almost primal.

Proof. Let I be a P -almost primary ideal of R . For every $a \in A(I)$, there exists $r \in R - I$ with $ra \in I - I^2$. Then as I is P -almost primary we get $a \in \sqrt{I} = P$. This implies that $A(I) \cup I^2 \subseteq P$. Now assume that $a \in P - I^2$. Suppose that m is the least positive integer for which $a^m \in I - I^2$. Then $aa^{m-1} \in I - I^2$ with $a^{m-1} \in R - I$ implies that a is not almost prime to I . Thus $P \subseteq A(I) \cup I^2$, and hence we have $P = A(I) \cup I^2$, that is I is P -almost primal. The proof of the last part is easy, because every almost prime ideal is almost primary.

The concept of weakly primary ideals introduced in [6]. Recall that a proper ideal Q of R is said to be weakly primary if $0 \neq ab \in Q$ implies $a \in Q$ or $b \in \sqrt{Q}$. In this case $P = \sqrt{Q}$ is a weakly prime ideal of R , and Q is called P -weakly primary. The following result provides the relations between almost primal and weakly primal ideals as well as between almost primary and weakly primary ideals.

Theorem 12. Let I and P be proper ideals of R with $I \subseteq P$. Then

- (1) I is a P -almost primal ideal of R if and only if I / I^2 is a P / I^2 -weakly primal ideal of R / I^2 .
- (2) I is a P -almost primary ideal of R if and only if I / I^2 is a P / I^2 -weakly primary ideal of R / I^2 .

Proof. Set $\bar{R} = R / I^2, \bar{I} = I / I^2, \bar{P} = P / I^2$ and denote by \bar{a} the coset a / I^2 for each $a \in R$.

(1) First assume that I is a P -almost primal ideal of R . Suppose that $\bar{a} \in \bar{R}$ is not weakly prime to \bar{I} . Then $\bar{a} \neq \bar{0}$, and $\bar{0} \neq \bar{a}\bar{b} \in \bar{I}$ for some $\bar{b} \in \bar{R} - \bar{I}$. This implies that $a \notin I^2$ and $ab \in I - I^2$ with $b \in R - I$, that is a is not almost prime to I . So that $a \in P - I^2$, that is $\bar{a} \in \bar{P} - \{\bar{0}\}$. Therefore $w(\bar{I}) \cup \{\bar{0}\} \subseteq \bar{P}$, where $w(\bar{I})$ denotes the set of all $\bar{a} \in \bar{R}$ such that \bar{a} is not weakly prime to \bar{I} . Now pick an element $\bar{c} \in \bar{P} - \{\bar{0}\}$. Then there exists $a \in I^2$ such that $c + a \in P$. This implies that $c + a$ is not almost prime to I . So $(c + a)d \in I - I^2$ for some $d \in R - I$. Since $(c + a) - c = a \in I^2$, we have $\overline{c+a} = \bar{c}$. Then $\bar{0} \neq \bar{c}\bar{d} \in \bar{I}$ with $\bar{d} \in \bar{R} - \bar{I}$. Therefore \bar{c} is not weakly prime to \bar{I} , that is $\bar{P} \subseteq w(\bar{I}) \cup \{\bar{0}\}$, and hence we have $\bar{P} = w(\bar{I}) \cup \{\bar{0}\}$. Consequently \bar{I} is a \bar{P} -weakly primal ideal of \bar{R} . Conversely, if \bar{I} is a \bar{P} -weakly primal ideal of \bar{R} , it is straight forward as above to show that $A(I) = P - I^2$. This implies that I is a P -almost primal ideal of R .

(2) Assume that I is P -almost primary in R . Let $\bar{a}\bar{b} \in \bar{R}$ be such that $0 \neq \bar{a}\bar{b} \in \bar{I}$. Then $ab \in I - I^2$ and I almost primary gives either $a \in I$ or $b \in \sqrt{I} = P$. Therefore, either $\bar{a} \in \bar{I}$ or $\bar{b} \in \bar{P}$. Hence \bar{I} is weakly primary. Conversely, assume that \bar{I} is \bar{P} -weakly primary. Let $a, b \in R$ be such that $ab \in I - I^2$. Then $0 \neq \bar{a}\bar{b} \in \bar{I}$. Since \bar{I} is \bar{P} -weakly primary, we get either $\bar{a} \in \bar{I}$ or $\bar{b} \in \sqrt{\bar{I}} = \bar{P}$. So either $a \in I$ or $b \in P$ as needed.

Proposition 13. Let J and P be ideals of R with $J \subseteq P^2$. Then P is an almost prime ideal of R if and only if P / J is an almost prime ideal of R / J .

Proof. If P is almost prime, then P / J is almost prime by [7, Proposition 15]. Now assume that P / J is almost prime, and let $a, b \in R$ are such that $ab \in P - P^2$.

Then $(a + J)(b + J) \in (P / J) - (P / J)^2$ since $J \subseteq P^2$. As P / J is almost prime, it follows that either $a + J$ or $b + J$ is in P / J . Thus either a or b is in P , that is P is almost prime.

Theorem 14. Let I and J be ideals of R with $J \subseteq I^i \subseteq P$ for $i = 1, 2$. Then I is an almost primal ideal of R if and only if I / J is an almost primal ideal of R / J .

Proof. First assume that I is a P -almost primal ideal of R . Then, by Lemma 4, and [7, Proposition 15], P / J is an almost prime ideal of R / J . We show that I / J is a P / J -almost primal ideal of R / J . We claim that $R / J = A(I / J) \cup (I / J)^2$. Let $A + J \in P / J - (I / J)^2$. Then $a \in P - I^2$, that is a is not almost prime to I . Hence there exists $r \in R - I$ with $ra \in I - I^2$. So we have $(r + J)(a + J) \in (I / J) - (I / J)^2$ with $r + J \in R / J - I / J$. Hence $a + J$ is not almost prime to I / J . Now assume that $b + J$ is not almost prime to I / J . Then $b + J \notin (I / J)^2$ and there exists $r + J \in R / J - I / J$ with $rb + J = (r + J)(b + J) \in I / J - (I / J)^2$. Hence $rb \in I - I^2$ with $r \notin I$, that is b is not almost prime to I . Hence $b \in P - I^2$ and so b

$+J \in P/J - (I/J)^2$. We have already shown that $P/J - (I/J)^2$ consists exactly of elements of R/J that are not almost prime to I/J . Hence I/J is almost primal with the adjoint ideal P/J . Conversely, assume that I/J is a P/J -almost primal ideal of R/J . We will show that I is a P -almost primal ideal of R . By Lemma 4, P/J is an almost prime ideal of R/J . Also $J \subseteq I^2 \subseteq P^2$ by our assumption. Hence P is an almost prime ideal of R by Proposition 13. It suffices to show that $P - I^2$ consists exactly of elements of R that are not almost prime to I . Let $a \in P - I^2$. Then $a + J \in (P/J) - (I/J)^2$, that is $a + J$ is not almost prime to I/J . Hence there exists $r + J \in (R/J) - (I/J)$ such that $(a + J)(r + J) \in (I/J) - (I/J)^2$. therefore $ar \in I - I^2$ with $r \in R - J$. This implies that a is not almost prime to I . Now assume that b is not almost prime to I . Then there exists $r \in R - I$ with $rb \in I - I^2$. Consequently $(r + J)(b + J) = rb + J \in (I/J) - (I/J)^2$ with $r + J \notin I/J$. Hence $b + J$ is not almost prime to I/J , that is $a + J \in (P/J) - (I/J)^2$; so $a \in P - I^2$. It follows that $P - I^2$ is exactly the set of elements of R that are not almost prime to I . Hence I is P -almost primal.

REFERENCES

- [1] Fuchs L., On primal ideals, Proc. Amer. Math. Soc., 1950; **1**: 1-6.
- [2] Ebrahimi Atani S. and Yousefian Darani A., Notes on the primal submodules, Chiang Mai J. Sci., 2008; **35(3)**: 399-410.
- [3] Fuchs L. and Mosteig E., Ideal theory in Prufer domains, J. Algebra, 2002; **252**: 411-430.
- [4] Ebrahimi Atani S. and Yousefian Darani A., On weakly primal ideals(I), Demonstratio Mathematica, 2007; **40**: 23-32.
- [5] Bhatwadekar S.M. and Sharma P.K., Unique factorization and birth of almost primes, Comm. Algebra, 2005; **33**: 43-49.
- [6] Ebrahimi Atani S. and Farzalipour F., On weakly primary ideals, Georgian Math. J., 2005; **12**: 423-429.
- [7] Anderson D.D. and Bataineh M., Generalizations of prime ideals, Comm. Algebra, 2008; **36**: 686-696.