



On the Graded Primary Avoidance Theorem

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ABSTRACT

Let G be a monoid with identity e , and let R be a G -graded commutative ring. Here we study the graded primary sub-modules of a graded R -module. While the bulk of this work is devoted to investigate the graded primary avoidance theorem for modules.

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1. INTRODUCTION

The prime avoidance theorem for modules has been introduced and studied by Chin-Pi Lu in [2]. Here we study the graded primary avoidance theorem for modules. A number of results concerning the graded primary avoidance theorem are given.

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary monoid with identity. A commutative ring R with non-zero identity is G -graded if it has a direct sum decomposition (as an additive group) $R = \bigoplus_{g \in G} R_g$ such that $1 \in R_e$ and for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. Let I be an ideal of R . For $g \in G$, let $I_g = I \cap R_g$. Then I is a graded ideal of R if $I = \bigoplus_{g \in G} I_g$. In this case I_g is called the g -component of I for $g \in G$.

If R is G -graded, then an R -module M is said to be G -graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be *homogeneous* element. A submodule $N \subseteq M$, where M is G -graded,

is called G -graded R -submodule of M if $N = \bigoplus_{g \in G} (N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G -graded module with g -component $(M/N)_g = (M_g + N)/N$. for $g \in G$. Clearly, 0 is a graded submodule of M . Also, we write $\mathfrak{h}(R) = \bigcup_{g \in G} R_g$ and $\mathfrak{h}(M) = \bigcup_{g \in G} M_g$. A graded ideal I of R is said to be *graded prime ideal* if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in R$. The *graded radical* of I , denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A graded ideal I of R is said to be *graded primary ideal* if $I \neq R$; and whenever $a, b \in \mathfrak{h}(R)$ with $ab \in I$, then $a \in I$ or $b \in Gr(I)$. In this case, $Gr(I) = P$ is a graded prime ideal of R , and we say that I is a graded P -primary ideal of R . (see [7, Lemma 1.8]).

2. GRADED PRIMARY SUB-MODULES

The graded primary and primary submodules are different concepts (see [7, Example 1.6]). So we recall from [3, 4] the definitions of such submodules. Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . N is a graded prime submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $a m \in N$, then either $m \in N$ or $a \in (N :_R M)$. N is a graded primary submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $a m \in N$, then either $m \in N$ or $a^k \in (N :_R M)$ for some k .

The following lemma is known, but we write it here for the sake of references.

Lemma 2.1 Let I, J be graded ideals of a G -graded ring R , M a graded R -module and N, K graded R -submodules of M . The following hold:

(i) $(N:M, IJ, I+J$ and $I \cap J$ are graded ideals of R .

(ii) If $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M .

Lemma 2.2 Let R be a G -graded ring and N, K, L graded R -submodules of a graded R -module M with $N \subseteq K \cup L$. Then $N \subseteq K$ or $N \subseteq L$.

Proof. Suppose $N \subseteq K \cup L$ but $N \not\subseteq K$ and $N \not\subseteq L$. Then there are homogenous elements $x_g \in N - K$ and $y_h \in N - L$ so $t = x_g + y_h \in N$; hence either $t \in K$ or $t \in L$ which is a contradiction (since K, L are graded). Thus $N \subseteq K$, or $N \subseteq L$.

Assume that R is a G -graded ring and let N, N_1, \dots, N_n be graded submodules of a graded R -module M . We call a covering $N \subseteq N_1 \cup N_2 \cup \dots \cup N_n$ efficient if no N_k is superfluous. Analogously, we shall say that $N = N_1 \cup N_2 \cup \dots \cup N_n$ is an *efficient union* if none of the N_k may be excluded. Any cover or union consisting of submodules of M can

be reduced to an efficient one, called an *efficient reduction*, by deleting any unnecessary terms. A covering of a graded submodule by two graded submodules of a graded module is never efficient by Lemma 2.2. Thus, $N \subseteq N_1 \cup N_2 \cup \dots \cup N_n$ may possibly be an efficient covering only when $n > 2$ or $n = 1$ [5].

Lemma 2.3. Assume that R is a G -graded ring and let $N = N_1 \cup N_2 \cup \dots \cup N_n$ be an efficient union of graded submodules of a graded R -module M for $n > 1$. Then

$$\bigcap_{j=k} N_j = \bigcap_{j=1} N_j \text{ for all } k.$$

Proof. It suffices to show that, say $k = 1$,

$\bigcap_{j=2} N_j \subseteq \bigcap_{j=1} N_j$. Let $m = m_{g_1} + \dots + m_{g_s} \in \bigcap_{j=2} N_j$ with $m_{g_i} \neq 0$ and let n_h be a homogenous element of $N - \bigcup_{j=2} N_j$, so $n_h \in N_1$; hence N_2, \dots, N_n graded gives $m + n_h \in N - \bigcup_{j=2} N_j$. It follows that $m \in N_1$ since N_1 is graded, as needed.

Lemma 2.4. Let I, I_1, \dots, I_k be graded ideals of a G -graded ring R with I graded primary such that $I_i \not\subseteq Gr(I)$. Then $I_1 \dots I_k \not\subseteq Gr(I)$.

Proof. Suppose not. By assumption, there are homogenous elements $x_{g_i} \in I_i - Gr(I)$ ($i = 1, \dots, k$) such that $x_{g_1} x_{g_2} \dots x_{g_k} \in Gr(I)$, so $Gr(I)$ graded prime (see [4, Lemma 1.8]) gives $x_{g_j} \in Gr(I)$ for some j which is a contradiction, as required.

Proposition 2.5. Assume that R is a G -graded ring and let $N \subseteq N_1 \cup \dots \cup N_n$ be an efficient covering consisting of graded submodules of a graded R -module M . If $Gr(N_j : M) \not\subseteq Gr(N_k : M)$ for every $j \neq k$, then no N_k is a graded primary submodule of M .

Proof. Since $N \subseteq N_1 \cup \dots \cup N_n$ is an efficient covering, $N = (N \cap N_1) \cup \dots \cup (N \cap N_n)$ is an efficient union. Hence, there exists a

homogeneous element $e_k \in N - N_k$ for every $j \in \{1, 2, \dots, n\}$. Moreover, $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$ by Lemma 2.4. If $j \neq k$, then $Gr(N_j : M) \not\subseteq Gr(N_k : M)$ so that there exists a homogeneous element $x_{g_i} \in Gr(N_j : M)$, but $x_{g_i} \notin Gr(N_k : M)$. Now, suppose that N_k is a graded primary submodule. Then $(N_k : M)$ is a graded primary ideal by [1, Proposition 2.7], therefore, the homogeneous element $x_h = \prod_{j \neq k} x_{g_j} \in Gr(N_j : M)$, but $x_h \in Gr(N_k : M)$ by Lemma 2.5. Then there exist positive integers s_1, \dots, s_n with $x_{g_1}^{s_1} \in (N_1 : M)$, $x_{g_2}^{s_2} \in (N_2 : M)$, \dots , $x_{g_n}^{s_n} \in (N_n : M)$. Let $s = \max\{s_1, \dots, s_n\}$. Then $x_h^s \in (N_j : M)$ for every $j \neq k$ but $x_h^s \notin (N_k : M)$. Hence, $x_h^s e_k \in N \cap N_j$ for every $j \neq k$, but $x_h^s e_k \notin N \cap N_k$, (if $x_h^s e_k \in N \cap N_k$ then N_k graded primary gives $x_h \in (N_k : M)$; hence this is a contradiction to $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$, and the proof is complete.

Theorem 2.6. [The Graded Primary Avoidance Theorem] Let M be a graded module over a G -graded ring R , N_1, \dots, N_n a finite number of graded submodules of M and N a graded submodule of M such that $N \subseteq N_1 \cup \dots \cup N_n$. Assume that at most two of the N_k 's are not graded primary, and that $Gr(N_j : M) \not\subseteq Gr(N_k : M)$ whenever $j \neq k$. Then $N \subseteq N_k$ for some k .

Proof. We may assume that the covering is efficient since the hypothesis remains valid after reduction to an efficient covering. Then $n \neq 2$. Also $n \leq 2$ by Proposition 2.6. Hence $n = 1$.

Theorem 2.7 [The Graded Prime Avoidance Theorem] Let M be a graded module over a G -graded ring R , N_1, \dots, N_n

a finite number of graded submodules of M and N a graded submodule of M such that $N \subseteq N_1 \cup \dots \cup N_n$. Assume that at most two of the N_k 's are not graded prime, and that $(N_j : M) \not\subseteq (N_k : M)$ whenever $j \neq k$. Then $N \subseteq N_k$ for some k .

Proof. By [1, Proposition 2.3], a graded submodule of a graded module over a G -graded ring R is prime if and only if it is prime in the graded sense. Now the assertion follows from [2, Theorem 2.3].

Theorem 2.8 [The Graded Primary Avoidance Theorem] Let R a G -graded ring R , I_1, \dots, I_n a finite number of graded ideals of R and I a graded ideal of R such that $I \subseteq I_1 \cup \dots \cup I_n$. Assume that at most two of the I_k 's are not graded primary, and that $Gr(I_j) \not\subseteq Gr(I_k)$ whenever $j \neq k$. Then $I \subseteq I_k$ for some k .

Proof.

This follows from Theorem 2.6.

Lemma 2.9

Assume that R is a G -graded ring and let N, K be graded submodules of a graded R -module M with $K \subseteq N$. Then N is a graded primary submodule of M if and only if N/K is a graded primary submodule of M/K .

Proof. Let N be a graded primary submodule of M and $r_g(m_h + K) \in N/K$ with $m_h + K \notin N/K$ where $r_g \in h(R)$ and $m_h \in h(M)$. It follows that $r_g m_h \in N$ with $m_h \notin N$, so hence N graded primary gives $r_g^s M \subseteq N$ for some s ; hence $r_g^s(M/K) \subseteq N/K$. Thus M/K is graded primary. The remaining implication is similar.

Theorem 2.10 Let N be a graded submodule of a graded module M over a G -graded ring R . If the graded primary avoidance theorem hold for M , then the graded primary avoidance theorem holds for M/N .

Proof. This follows from Lemma 2.9.

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