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Contributed Paper

Some Geometry of Quotient Spaces on C(0,1)

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ABSTRACT

In this paper we investigate some geometry of the quotient space X/M where X is a Banach space and M is its closed subspace. It is shown that the quotient space X/M is nearly uniform convexity whenever X is nearly uniform convexity. It is noted that X/M has the fixed point property.

Keywords : geometric properties of Banach space, quotient space.

1. INTRODUCTION

Geometry of Banach space is an important topic in functional analysis and plays an important role in the theory of approximation and optimization. The property of uniform rotundity ensures, for example, the existence and unity of nearest points in best approximation problems. Among geometric properties, nearly uniform convexity (NUC) and uniform Kadac-Klee (UKK) are also important. Both of them follow from the uniform convexity (UC) and that (UKK) implies property (H), and (NUC) implies (UKK).

Summarizing the above discussion we have

$$(UC) \Longrightarrow (NUC) \Longrightarrow (UKK) \Longrightarrow \text{property (H)}$$

Now we introduce the basic notations and definitions.

Let $(X, \|\cdot\|)$ be a real Banach space, and let B(X) (resp. S(X)) be the closed unit ball(resp. the unit sphere) of X. For any subset A of X, we denote by conv(A) (resp. $\overline{\text{conv}}(A)$) the convex hull (resp. the closed convex hull) of A and X^* be its dual space. Clarkson [1] introduced the concept of uniform convexity.

The norm $\|\cdot\|$ is called *uniformly convex* (write (UC)) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ the inequality

$$\|x-y\| > \varepsilon \text{ implies } \|(x+y)/2\| < 1 - \delta .$$

A Banach space X is said to have the *Kadac-Klee property* (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $\{x_n\} \subset X$ is said to be ε - *separated* sequence for some $\varepsilon > 0$ if $\text{sep}(\{x_n\}) = \inf \{\|x_n - x_m\| : n \neq m\} > \varepsilon$

A Banach space is said to be *uniformly Kadac-Klee property* (write (UKK)) if for every $\varepsilon > 0$ there exists $\delta > 0$ for every sequence $\{x_n\}$ in S(X) with $\text{sep}(\{x_n\}) > \varepsilon$ and $x_n \xrightarrow{w} x$ ($x_n \xrightarrow{w} x$ if, for every $f \in X^*$, $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$), we have $\|x\| < 1 - \delta$. It is proved that every (UKK) Banach space has property (H) [2].

A Banach space is said to be *nearly uniformly convex* (write (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $\{x_n\} \subseteq B(X)$ with $\text{sep}(\{x_n\}) > \varepsilon$, we have

$$\text{conv}(\{x_n\}) \cap ((1 - \delta) B(X)) \neq \Phi$$

Huff [3] proved that every (NUC) Banach space is reflexive and it has a property (H) and he proved that X is (NUC) if and only if X is reflexive and (UKK). Hence we have

$$\text{(NUC)} \Leftrightarrow \text{(UKK) and reflexive.} \qquad (1)$$

Recall that a Banach space X is said to have the *weakly uniformly Kadac-Klee property* (write (WUKK)) if there exists $\varepsilon \in (0,1)$ and $\delta > 0$ such that $\|x\| \le 1-\delta$ whenever x is a weak limit of some sequence $\{x_n\}$ in the unit ball with $\liminf_{n\to\infty} \|x_n - x\| \ge \varepsilon$. Clearly the UKK property of a Banach space X implies that X has (WUKK).

Recall the quotient space X/M where M is a subspace of a vector space X is the vector space whose underlying set is the collection $\{x + M : x \in X\}$.

If M is a closed subspace of a Banach space X. It is well known that X/M equiped with the norm $\|[x]\| = \inf\{\|y\| : y \in [x]\}$, where $[x] = \{y \in X : y - x \in M\}$, is also a Banach space.

Remark1. Since $\|[x]\| = \inf\{\|y\| : y \in [x]\}$, it follows immediately that
$\|[x]\| \le \|x\|$, for any $[x] \in X/M$.

Recall *a fixed point* of a mapping $T:X \to X$ of a set X into itself is an $x \in X$ such that $Tx = x$

Let K be a nonempty bounded closed convex subset of a Banach space X. A mapping $T:K \to K$ is said to be *nonexpansive* whenever the inequality

$$\|Tx - Ty\| \le \|x - y\|$$

holds for every $x, y \in K$. We say that K has the FPP(resp. fixed point property) if every nonexpansive mapping has a fixed point. And we will say that X has the FPP if every weakly compact convex $K \subseteq X$ has the FPP.

Garcia – Falset [4] proved that the Banach space X and X^* have the FPP whenever X is reflexive and has (WUKK).

2. Main Results

Let X = C[0,1] and in order to prove the main result the following lemma is needed.

Lemma1. If X is a reflexive Banach space and M is its closed subspace, then for any $\varepsilon > 0$ and $[x_n]$, $[x]$ are in B(X/M) ,with $sep(\{[x_n]\}) > \varepsilon$ and $[x_n] \xrightarrow{w} [x]$ we have that $x_n \xrightarrow{w} x \in B(X)$ and $sep(\{x_n\}) > \varepsilon$.

Proof. Let $\{[x_n]\}$ be a sequence in B(X/M) and $[x] \in B(X/M)$ such that $sep(\{[x_n]\}) > \varepsilon$ and $[x_n] \xrightarrow{w} [x]$. Since X is reflexive, there is $x' \in B(X)$ such that $x_{n_k} \xrightarrow{w} x'$. Therefore $[x'] = [x]$, which shows that $x_{n_k} \xrightarrow{w} x \in B(X)$. By definition of weak convergence we get for every $f \in X^*$, $f(x_{n_k}) \to f(x)$. Since $\{f(x_{n_k})\}$ is a sequence of numbers, it is bounded, say

$$|f(x_{n_k})| < C$$

where C is a constant depending on f.

Let $\varepsilon_1 > 0$, we get that there exists an positive integer N_1 such that

$$|f(x_{n_k}) - f(x)| < \frac{\varepsilon_1}{2} \; ; \forall k > N_1$$

Since X is complete, there is an positive integer N_2 such that

$$\|x_n - x_{n_k}\| < \frac{\varepsilon_1}{2C} \; ; \forall n, k > N_2$$

Let $N = \max\{N_1, N_2\}$, we obtain that for all $n, k > N$

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n) - f(x_{n_k}) + f(x_{n_k}) - f(x)| \\ &\le |f(x_n) - f(x_{n_k})| + |f(x_{n_k}) - f(x)| \\ &\ge \|f\| \, \|x_n - x_{n_k}\| + |f(x_{n_k}) - f(x)| \\ &< C\left(\frac{\varepsilon_1}{2C}\right) + \left(\frac{\varepsilon_1}{2}\right) = \varepsilon_1 \end{aligned}$$

Since $f \in X^*$ is abitrary, this shows that $x_n \xrightarrow{w} x \in B(X)$. And by Remark 1, that is $sep(\{x_n\}) \ge sep(\{[x_n]\}) > \varepsilon$.

Theorem1. If a Banach space X is (NUC) then X/M is also (NUC), where M is a closed subspace of X.

Proof. Let M be a closed subspace of X and assume that X is (NUC). It follows from (1) that X is reflexive and has property (UKK). From the reflexivity of X, X/M is also reflexive [5]. We need only to show that X/M has property (UKK).

Let $\varepsilon > 0$ and we want to show that there exists $\delta > 0$ such that whenever a sequence $\{[x_n]\}$ in B(X/M) and a point $[x] \in B(X/M)$ with $sep(\{[x_n]\}) > \varepsilon$ and $[x_n] \xrightarrow{w} [x]$, we have $\|[x]\| < 1 - \delta$.

Since X has (UKK), there exist $\delta > 0$ such that if $\{x_n\}$ is a sequence in B(X) with $sep(\{x_n\}) > \varepsilon$ and $x_n \xrightarrow{w} x \in B(X)$, then $\|x\| < 1 - \delta$.

Assume that $\{[x_n]\}$ in B(X/M) and a point $[x] \in B(X/M)$ with $sep(\{[x_n]\}) > \varepsilon$ is such that $[x_n] \xrightarrow{w} [x]$. By Lemma1 we get $x_n \xrightarrow{w} x \in B(X)$ and $sep(\{x_n\}) > \varepsilon$, hence there exists $\{x_n\} \in B(X)$ and we get that $\|x\| < 1 - \delta$. And by Remark 1 we have $\|[x]\| \leq \|x\| < 1 - \delta$, which yield that X/M is (UKK). Together with X/M is reflexive we obtain that X/M is (NUC).

Corollary 1. If a Banach space X is NUC then X/M and $(X/M)^*$ have the FPP, where M is a closed subspace of X.

Proof. From the assumption and Theorem1 we have that X/M is reflexive and has UKK property. Also X/M has WUKK property. By Gracia - Falset (see [4]) we have, X/M and $(X/M)^*$ have the FPP.

REFFERENCE

[1] Clarkson J. A., Uniformly convex spaces, *Trans. Amer. Math.*, 1936; **40** :396-414.

[2] Kantorovich L.V. and Akilov G.P., *Functional Analysis.*,Moscow : Nauka, 1978.

[3] Huff R., Banach spaces which are nearly uniformly convex, *Rocky Mountian J. Math.*, 1985; **10** : 473-749.

[4] Garcia-Falset J., The Fixed Point Property in Banach Spaces with the NUS-property, *J.Math. Anal. Appl.*, 1997; **215** : 532-542.

[5] Megginson E., *An Introduction to Banach Space Theory, Graduate Texts in Math.,* 183., New York : Springer-Verlag,1991.