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Contributed Paper

Dunwoody Parameters and Abstract Groups

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ABSTRACT

We have proved in this paper that cyclically presented groups obtained using the word w connected with Dunwoody parameters $(a,b,c,r)=(1,0,c,2)$ are associated with the abstract groups $S\left((d+1)/2,d\right)$ when c is odd and $d=2a+b+c$. We also gave two different examples to illustrate how to apply our result to particular problems in application.

Keywords : abstract groups, cyclically presented groups, dunwoody parameters.

1. INTRODUCTION AND DEFINITIONS

Let F_n be the free group on free generators $x_0, x_1, x_2, \ldots, x_{n-1}$. Let $\theta : F_n \to F_n$ be the automorphism such that $\theta(x_i) = x_{i+1}$, $i=0,1,\ldots,n-2$, $\theta(x_{n-1}) = x_0$.

For $w \in F_n$, $G_n(w)$ is defined as $G_n(w) = F_n / R$ where R is the normal closure in F_n of the set $\{w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w)\}$ [1].

For a reduced word $w \in F_n$, the cyclically presented group $G_n(w)$ is given by $G_n(w) = \langle x_0, x_1, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w)\rangle$ [2].

Definition 1.1: A group G is said to have a cyclic presentation if $G \cong G_n(w)$ for some n and w [3].

Definition 1.2: The Sieradski group $S(r,n)$ is defined by the cyclic presentation

$$\langle x_1, x_2, \ldots, x_n : x_i x_{i+2} = x_{i+1} \rangle,$$

where all indices are in modulo n.

Definition 1.3: Abstract group is defined by the cyclic presentation $S(r,n) = \langle x_1, x_2, \ldots, x_n : x_i x_{i+2} \ldots x_{i+2r-2} = x_{i+1} x_{i+3} \ldots x_{i+2r-3} \rangle$ (indices are again in modulo n) for any two positive integers r and $n \geq 2$. For $r=2$, these $S(r,n)$ present the Sieradski groups [4].

Let a,b,c,n be integers such that $n>0$, $a,b,c \geq 0$ and $a+b+c>0$. Let $\bar{\tau}(a,b,c)$ be the graph shown in Figure 1. This is an infinite graph with an automorphism θ such that $\theta(u_n)=u_{n+1}$ and $\theta(v_n)=v_{n+1}$. The labels indicate the number of edges joining a pair of vertices. Thus, there are a edges joining u_1 and u_2. We see that the $\bar{\tau}(a,b,c)$ is d-regular where $d=2a+b+c$. Let $\tau_n = \tau_n(a,b,c)$

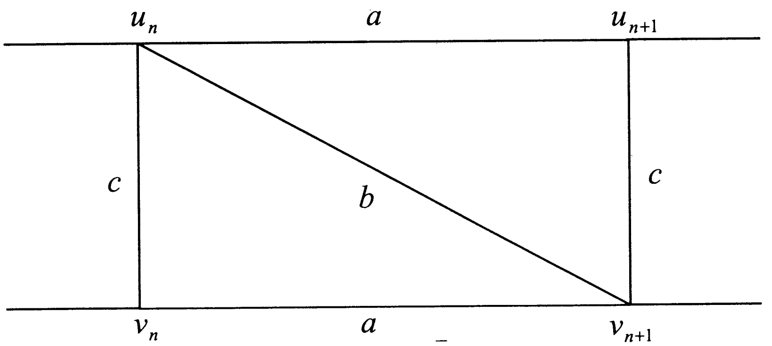

Figure 1. Graph of $\bar{\tau}(a,b,c)$.

denote the graph obtained from $\bar{\tau}(a,b,c)$ by identify-ing all edges and vertices in each orbit of θ^n. Thus τ_n has $2n$ vertices [5].

We say that the 6-tuple (a,b,c,r,s,n) has property M if it corresponds to the Heegaard diagram of a 3-manifold. An algorithm determining which 6-tuples have property M is now described. Put $d = 2a+b+c$ and let $X = \{-d,-d+1,\ldots,-1,1,2,\ldots,d\}$.

Let α, β be the permutations of X defined as follows:

$$\alpha = (1,d)(2,d-1),\ldots,(a,\ d-a+1)(a+1,-a-c-1)(a+2,\ -a-c-2)\ldots$$
$$(a+b,-a-c-b)(a+b+1,-a-1)(a+b+2,-a-2)\ldots(a+b+c,-a-c)(-1,-d)$$

and

$$\beta(j) = \begin{cases} -(j+r) & , \text{ if } \ j>0 \text{ and } j+r\le d \text{ or } j<0 \text{ and } j+r<0 \\ -(j+r-d) & , \text{ if } \ j+r\ge 0 \end{cases}$$

Now we give the following theorem to demonstrate how the 6-tuples (a,b,c,r,s,n) have property M. Details and the proof of this theorem can be found in [5].

Theorem 1: Let $d = 2a+b+c$ be odd . The 6-tuple (a,b,c,r,s,n) has property M if and only if the following two conditions hold simultaneously:

i. $\alpha\beta$ has two cycles of length d

ii. $ps+q=0 \pmod n$

where p is the difference between the number of arrows pointing down the page and the number of arrows pointing up, whereas q is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by $\alpha\beta$. The entires in the first cycle of $\alpha\beta$ contain one vertex from each line segment of the diagram. There exist an integer s such that $ps+q\equiv 0$. The first cycle of $\alpha\beta$ and the value of s can also be used to calculate the word w of the corresponding cyclic presentation.

2. Materials and Methods

We can now state our theorem:

Theorem 2: Cyclically presented groups obtained by using the word w constituted as connected with Dunwoody parameters $(a,b,c,r)=(1,0,c,2)$ are associated with the abstract groups $S((d+1)/2,d)$ when c is an odd positive integer and $d=2a+b+c$.

Proof: It was proved that if $d>3$ and $d=2a+b+c$ is odd, the terms in the first cycle of $\alpha\beta$ for 6 -tuple $(1,0,c,n,r,s)$ have of the following form $(1,-2,-4,\ldots,-d+1,-1,c,c-2,c-4,\ldots,3)$ and the word w constituted as connected with Dunwoody parameters $(1,0,c,n,r,s)$ has the following form [6]

$$w = x_{d-1}^{-1}x_{d-3}^{-1}x_{d-5}^{-1}\ldots x_{d-c}^{-1}x_0^{-1}x_1x_3x_5\ldots x_c \quad (2.1)$$

For this reduced word w, the cyclically presented group $G_d(w)$ is

$$\begin{aligned} G_d(w) &= G_d(x_{d-1}^{-1}x_{d-3}^{-1}x_{d-5}^{-1}\ldots x_{d-c}^{-1}x_0^{-1}x_1x_3x_5\ldots x_c) \\ &= \langle x_1,x_2,\ldots,x_d \mid x_ix_{i+2}\ldots x_{i+d-5}x_{i+d-3}x_{i+d-1} \\ &= x_{i+1}x_{i+3}x_{i+5}\ldots x_{i+c}\rangle, \end{aligned}$$

where subscripts are understood to be reduced modulo d to lie in the set $\{1,2,\ldots,d\}$. It can be easily seen that the groups $G_d(w)$ have exactly the same presentation as the abstract groups, where $S(r,n) = \langle x_1,x_2,\ldots,x_n : x_ix_{i+2}\ldots x_{i+2r-2} = x_{i+1}x_{i+3}\ldots x_{i+2r-3}\rangle$ (indices are in modulo n) for any two positive integers r and $n\ge 2$, given by definition 1.3. We get $x_{i+d-1} = x_{i+2r-2}$ from corresponding terms and also $i+d-1=i+2r-2$. Thus, $r=(d+1)/2$ and $n=d$.

According to these values of $r=(d+1/2)$ and $n=d$, the abstract group is $S((d+1)/2,d)$.

Particularly, when $d=3$ for $c=1$, the terms in the first cycle of $\alpha\beta$ for 4-tuple $(a,b,c,r)=(1,0,1,2)$ are $(1,-2,-4,\ldots,-d+1,-1)$.

Notice that when $d=3$, from Eq. (2.1), the defining word w can be written as $w=x_2^{-1}x_0^{-1}x_1$. For this reduced word w, the cyclically presented group $G_3(w)$ is

$$G_3(w) = G_3(x_2^{-1}x_0^{-1}x_1)$$
$$= < x_0, x_1, x_2 \mid x_i x_{i+2} = x_{i+1} > .$$

On the other hand, we also get $x_{i+2} = x_{i+2r-2}$ from corresponding terms and where $i+2 = i+2r-2$. Thus, $r=2$ and $n=d=3$.

According to these values of $r=2$ and $n=3$, the abstract group is found as $S(2,3)$. $S(2,3)$ is *Sieradski group*, a specific case of abstract groups for $r=2$, given by Definition 1.2. This completes proof.

Now we give two examples for two different values of the parameter c to illustrate how to apply Theorem 2 to particular problems.

Example 1: According to theorem 2, Dunwoody parameters $(a,b,c,r) = (1,0,c,2)$ for $c=1$ are $(1,0,1,2)$. Then for this particular values of the parameters, d can be calculated as $d = 2a+b+c = 3$. Hence, the 4-tuple which has property M is $(1,0,1,2)$.

Thus ,for example, the case illustrated in Figure 2 is

$$X = \{-1,-2,-3,\ 1,\ 2,\ 3\}$$
$$\alpha = (1,\ 3)(2,-2)(-1,-3)$$

for $j = 1,2,3$ and $r=2$,

$$\beta = (1,-3)(2,-1)(3,-2)$$

and therefore,

$$\alpha\beta = (1,-2,-1)(2,\ 3,-3).$$

In this case, since the 4-tuple $(1,0,1,2)$ has property M ,$\alpha\beta$ is the product of two disjoint cycles of length $d=3$. For $p=-1$ and $q=2, s=2$

$$2\ (0)\ +1 \rightarrow x_1^{-1}$$
$$2(-1)\ +1 \rightarrow x_{-1}^{-1}$$
$$2(-1)\ +2 \rightarrow x_0$$

We have $w = x_1^{-1}x_{-1}^{-1}x_0$ or $w = x_2^{-1}x_0^{-1}x_1$.

For this reduced word w, the cyclically presented group $G_3(w)$ is

$$G_3(w) = G_3(x_2^{-1}x_0^{-1}x_1)$$
$$= < x_0, x_1, x_2 \mid x_i x_{i+2} = x_{i+1} > .$$

We also get $x_{i+2} = x_{i+2r-2}$ from corresponding terms. Here $i+2 = i+2r-2$ holds. Thus, $r=2$ and $n=d=3$.

According to these values of $r=2$ and $n=3$, the abstract group is $S(2,3)$. $S(2,3)$ is *Sieradski group*, a specific case of abstract groups for $r=2$, given by Definition 1.2.

Example 2: According to theorem 2, Dunwoody parameters $(a,b,c,r) = (1,0,c,2)$ for $c=3$ are $(1,0,3,2)$. For $a=1$, $b=0$, $c=3$, $d = 2a+b+c = 5$. Hence, the 4-tuple which has property M is $(1,0,3,2)$. Let first compute the word w constituted as connected with Dunwoody parameters $(a,b,c,r) = (1,0,3,2)$.

Thus, for example, the case illustrated in Figure 3,

$$X = \{-1,-2,-3,-4,-5,1,2,3,4,5\}$$
$$\alpha = (1,5)(2,-2)(3,-3)(4,-4)(-1,-5)$$

for $j = 1,2,3,4,5$ and $r=2$,

$$\beta = (1,-3)(2,-4)(3,-5)(4,-1)(5,-2)$$

and thus,

$$\alpha\beta = (1,-2,-4,-1,3)(-3,-5,4,2,5).$$

In this case, since the 4-tuple $(1,0,3,2)$

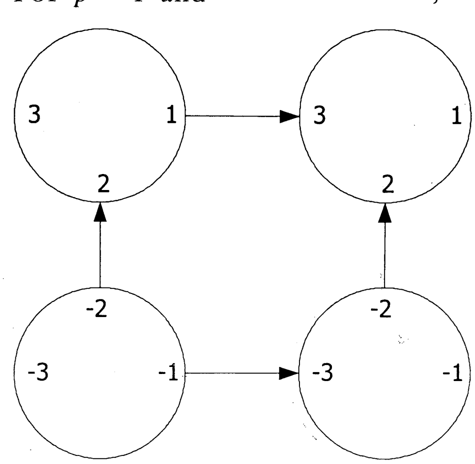

Figure 2. Heegaard diagram for 4-tuple $(1,0,1,2)$.

has property $M, \alpha\beta$ is the product of two disjoint cycles of length $d = 5$. For $p = -1$ and $q = 2$, $s = 2$

$$\begin{aligned} &2\ (0) + 1 \to x_1^{-1} \\ &2(-1) + 1 \to x_{-1}^{-1} \\ &2(-2) + 1 \to x_{-3}^{-1} \\ &2(-2) + 2 \to x_{-2} \\ &2(-1) + 2 \to x_0 \end{aligned}$$

We also have $w = x_1^{-1} x_{-1}^{-1} x_{-3}^{-1} x_{-2} x_0$ or $w = x_4^{-1} x_2^{-1} x_0^{-1} x_1 x_3$.

For this reduced word w, the cyclically presented group $G_5(w)$ is

$$\begin{aligned} G_5(w) &= G_5(x_4^{-1} x_2^{-1} x_0^{-1} x_1 x_3) \\ &= \langle x_0, x_1, x_2, x_3, x_4 \mid x_i x_{i+2} x_{i+4} = x_{i+1} x_{i+3} \rangle. \end{aligned}$$

Also, we get $x_{i+4} = x_{i+2r-2}$ from corresponding terms. Here $i + 4 = i + 2r - 2$ is satified. Thus, $r = 3$ and $n = d = 5$.

According to these values of $r = 3$ and $n = 5$, the abstract group is $S(3,5)$.

3. Results and Discussions

We have stated a proved theorem which relates cyclically presented groups obtained by using the word w constituted as connected with Dunwoody parameters $(a,b,c,r) = (1,0,c,2)$ and the abstract groups $S((d+1)/2, d)$ when c is odd and $d = 2a + b + c$. We also gave two different examples to show how to apply our result to particular problems in applications for smaller values of the parameter. We have seen that these examples confirm the result of our theorem.

On the other hand, calculation of these groups are extremely difficult and tedious for larger values of the parameter if it is performed by hand. A computer program could also be developed to overcome this issue.

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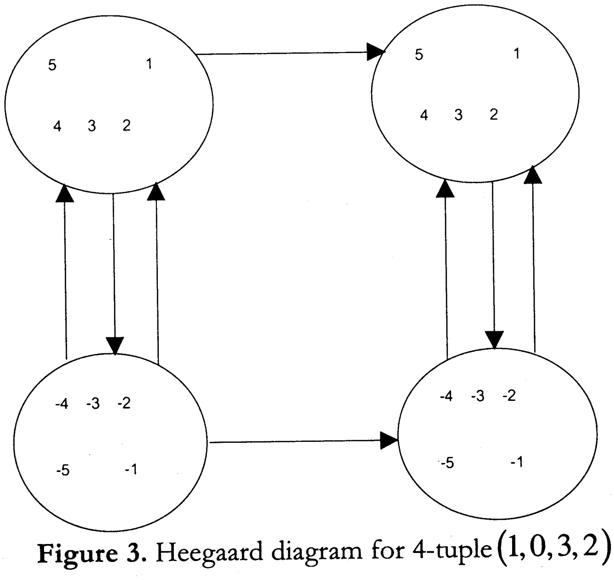

Figure 3. Heegaard diagram for 4-tuple $(1,0,3,2)$.