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On Prime and Primary Submodules

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ABSTRACT

This paper is devoted to study some properties of prime and primary submodules. First, by using the notion product of two submodules of a multiplication module we studied the radical of such submodules and related results. Second, we gave a condition which allows us to determine whether submodules of a module have prime radical.

Keywords : multiplication, primary, prime, secondary.

1. INTRODUCTION

To extend the concepts of prime ideal, primary ideal and the product of two ideals from the category of rings to the category of modules has stimulated several authors to show that many, but not all, of the results in the theory of rings are also valid for modules (see, for example, [1-3]).

Before we state some results let us introduce some notation and terminology. Throughout this note all rings will be commutative with identity and all modules will be unitary. If R is a ring and N is a submodule of an R-module M, the ideal $\{r\in R : rM\subset N\}$ will be denoted by $(N:M)$. Then $(0:M)$ is the annihilator of M. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I=(N:M)$ (see [2]).

Let M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1 I_2M$. Then by [1, theorem 3.4], the product of N and K is independent of presentation of N and K. Moreover, for $a,b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N\cap K$.

A proper submodule N of a module M over a ring R is said to be prime (resp. primary) if for each $r \in R$ the R-endomorphism of M/N produced by multiplication by r is either injective or zero (resp. nilpotent), so $(0:M/N)=P$ (resp. nilrad $(M/N)=P'$) is a prime ideal of R, and N is said to be P-prime submodule (resp. P'-primary submodule). So N is prime (resp. primary) in M if and only if whenever $rm\in N$, for some $r\in R$, $m\in M$, then $m\in N$ or $rM\subseteq N$ (resp. $m\in N$ or $r^kM\subseteq N$ for some k). Therefore, every prime is primary.

For any submodule N of an R-module M the radical, $Rad(N)$, N is defined to be the intersection of all prime submodules of M containing N and $Rad(N)=M$ if N is not contained in any prime submodule of M.

An R-module M is said to be secondary if $M\neq 0$ and for each $r\in R$ the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent, so nilrad $(M)=P$ is a prime ideal of R, and M is said to be P-

secondary (see [4]). Let N be an R-submodule of M. Then N is pure in M if $IN=N\cap IM$ for each ideal I of R.

The purpose of this paper is to explore some basic facts of these class of submodules. In section 2 we turn our attention to prove that some of the results in the theory of rings are also valid for multiplication modules. In section 3, various properties of prime submodules of a module are considered. For example, we show that over a one-dimensional domain R, for any submodule N of a secondary module M, $Rad(N)$ is a prime submodule.

2. THE PRODUCT OF MULTIPLICATION SUBMODULES

Note that it is possible that for a submodule N of a module M, has no presentation ideal. For example, if M is a vector space over an arbitrary field F with $\dim_F(M)=k\geq 2$, then M is finite length, so M is notherian, artinian, and pure-injective, but a proper subspace $N\neq 0$ of M has not any presentation. Moreover, M is not multiplication.

Proposition 2.1 Let N be a submodule of a faithful multiplication module M over a PID. Then

(i) N is multiplication.

(ii) $N\neq 0$ is a prime submodule of M if and only if N is a maximal submodule of M.

Proof. (i) There exists an ideal I of R such that $N = IM$, so I is a principal ideal of R, and hence it is multiplication. Now the assertion follows from [5, lemma 1.4] (note that over an integral domain, every faithful multiplication module is finitely generated).

(ii) This follows from [6, theorem 2.5 and corollary 2.11]. □

Lemma 2.2 Let N be a prime R- submodule of a finitely generated multiplication module M. Let $N_1, N_2, \ldots, N_n$ be submodules of M. Then the following statements are equivalent:

(*i*) $N \supseteq N_j$ for some j with $1 \leq j \leq n$.

(*ii*) $N \supseteq \bigcap_{i=1}^{n} N_i$.

(*iii*) $N \supseteq \prod_{i=1}^{n} N_i$.

Proof. It is clear that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ We have $N = PM$ and $N_i = I_iM$ for some ideals P and $I_i (1 \leq i \leq n)$ of R with P prime. Then $P = P+(0:M) \supseteq \prod_{i=1}^{n} I_i$ by [7, p.231 corollary], so $P \supseteq I_j$ for some j by [8, lemma 3.55], and hence $N \supseteq N_j$, as required. □

Corollary 2.3 Let N be a prime R-submodule of a finitely generated multiplication module M. Let $N_1, N_2, \ldots, N_n$ be submodules of M such that $N = \bigcap_{i=1}^{n} N_i$. Then $N = N_j$ for some j with $1 \leq j \leq n$.

Proof. This follows from lemma 2.2. □

Proposition 2.4 Let M be a non-zero multiplication R-module, and assume that M is annihilated by the product of finitely many (not necessarily distinct) maximal submodules of M. Then M is a noetherian R-module if and only if M is an artinian R-module.

Proof. Let $M_1, M_2, \ldots, M_n$ be maximal submodules of M such that

$$M_1M_2 \ldots M_nM = 0$$

There are maximal ideals $P_1, P_2, \ldots, P_n$ of R such that $M_i = P_iM$ with $(1 \leq i \leq n)$ by [6, theorem 2.5]. It follows that $P_1P_2 \ldots P_nM = 0$. Now the assertion follows from [8, theorem 7.30]. □

Lemma 2.5 Let N,K be submodules of a multiplication R-module M. Then

(*i*) If $N \subseteq K$, then $(K/N)^n = (K^n+N)/N$ for each positive integer n.

(*ii*) If $N \subseteq K$, then $Rad(K/N) = (Rad(K))/N$.

(*iii*) If M is finitely generated and N is a prime submodule of M, then $Rad(N^n) = N$ for each positive integer n.

Proof. (*i*) Since a quotient of any multiplication R-module is multiplication it follows from[1,Lemma2.6], that $(K/N)^n = ((K/N : M/N)M/N)^n$

$= (K:M)^n M/N = ((K:M)^n M+N)/N = (K^n + N)/N$.

(ii) We have $\mathrm{Rad}(K/N) = (\mathrm{Rad}(K/N : M/N)).(M/N) = (\mathrm{Rad}(K:M)).(M/N) = (\mathrm{Rad}K+N)/N = (\mathrm{Rad}K)/N$ by [6, theorem 2.12].

(iii) Since N is prime it follows that $(N : M) = P$ is a prime ideal of R. By [6, theorem 2.12] and [1, lemma 2.6], we have:

$\mathrm{Rad}(N^n) = (\mathrm{Rad}(N^n : M))M = (\mathrm{Rad}(N : M))^n)M = \mathrm{Rad}(P^n)M = PM = N$ □

Proposition 2.6 Let N be an R-submodule of a finitely generated multiplication module M such that $\mathrm{Rad}(N)=K$, a maximal submodule of M. Then N is a primary submodule of M. Moreover, K^n is primary for each positive integer n.

Proof. Since $N \subseteq \mathrm{Rad}(N) = K \neq M$, it is clear that N is a proper submodule of M. Let $r \in R$ and $m \in M$ be such that $rm \in N$ but $r \notin Rad(I)$ where $I = (N:M)$. By [6, theorem 2.5], there exists a maximal ideal P of R such that $K = \mathrm{Rad}(I)M = PM$. Therefore, $\mathrm{Rad}(I)$ is a maximal ideal of R, so we must have $\mathrm{Rad}(I)+Rr = R$, so that $I+Rr = R$ by [8, exer. 2.25]. Thus, there exist $a \in I$, $t \in R$ such that $1 = a+tr$, and hence $m = am+trm \in N$. Therefore, N is primary.

The last claim is now an immediate consequence, because $\mathrm{Rad}(K^n) = K$ for all positive integer n, by lemma 2.5 (iii) (since every maximal submodule is prime). □

Lemma 2.7 Let R be a commutative ring, M be an R-module, and N be a P-prime submodule of M. If K is a submodule M with $K \not\subset N$, then $N \cap K$ is a P-prime submodule of K.

Proof. This follows from [3, Lemma 1.6] □

Lemma 2.8 Let R be a commutative ring, M be a multiplication R-module, and P be a prime submodule of M. If N is a pure submodule M with $N \not\subset P$, then NP is a prime submodule of N.

Proof. By Lemma 2.7, it is enough to show that $N \cap P = NP$. There are ideals I and J of R such that $N = IM$ and $P = JM$, so $N \cap P = N \cap JM = JN = IJM = NP$ since N is pure in M, as required. □

Proposition 2.9 Let N be a primary R-submodule of a finitely generated multiplication module M. Then whenever $a,b \in M$ with $(Ra).(Rb) \subseteq N$ but $Ra \not\subset N$, then $Rb \subseteq Rad(N)$.

Proof. Assume that N is primary and let $(Ra).(Rb) \subseteq N$ with $Ra \not\subset N$. There exist ideals A, I and J of R such that $N = AM$, $Ra = IM$ and $Rb = JM$ respectively (see [6, proposition 1.1]). Since $Ra = IM \not\subset N$, there exist $r \in I$ and $m \in M-N$ such that $rm \notin N$. If $s \in J$ then $rsm = s(rm) \in IJM \subseteq N$, so $s \in Rad(A)$, and hence $J \subseteq Rad(A)$. Thus, $Rb = JM \subseteq Rad(A)M = Rad(N)$ by [6, theorem 2.12], as required. □

Lemma 2.10 Let N, K be submodules of a multiplication R-module M with K is finitely generated and $K \subseteq Rad(N)$. Then $K^n \subseteq N$ for some n.

Proof. Let $K = Ra_1+Ra_2+...+Ra_k$, and let $I_1,I_2,...,I_k$ be presentation ideals of $a_1,...,a_k$, respectively. There are positive integers $s_1,...,s_k$ such that $a_i^{s_i} = I_i^{s_i} M \subseteq N$ by [1, theorem 3.13] $(1 \leq i \leq k)$. Let $n = \max\{s_1,...,s_k\}$. It follows that

$K^n = (I_1M +...+ I_kM)^n = (I_1 +...+ I_k)^n M \subseteq N.$

□

Definition 2.11 Let M be a multiplication R-module. A zero divisor in M is an element $a \in M$ for which there exists $b \in M$ with $b \neq 0_M$ such that $a.b = Ra.Rb = 0_M$.

Theorem 2.12 Let N be a primary R-submodule of a finitely generated multiplication module M. Then the module M/N has the property that every zero divisor in M/N is nilpotent.

Proof. Suppose that N is primary. Let $a \in M$ be such that the element $\bar{a} = a+N$ in M/N is

zero divisor, so that there exists $b \in M$ such that $\bar{b} = b+N \neq 0_{M/N}$ but $\bar{a}.\bar{b} = 0$. Let I and J be presentation ideals $\bar{a}$ and $\bar{b}$, respectively (since a quotient of any multiplication R-module is multiplication). Clearly, $(Ra : M) = (R\bar{a} : M/N)$ and $(Rb : M) = (R\bar{b} : M/N)$. Then $(R\bar{a}).(R\bar{b}) = (IJ)M/N = N$, so $Ra.Rb = (IM).(JM) \subseteq N$ with $Rb \not\subset N$. Then $Ra \subseteq \text{Rad}(N)$ by proposition 2.9, and it follows from [1, theorem 3.13] that $(IM)^k = (Ra)^k \subseteq N$. Therefore we have $(\bar{a})^k = 0$, as needed. □

Theorem 2.13 Let M be a finitely generated faithful multiplication over a commutative ring R. Then R is an integral domain if and only if, whenever $N.K = 0$, then either $N = 0$ or $K = 0$ for all submodules N and K of M.

Proof. Assume first that R is an integral domain and let N and K be submodules of M. There exist ideals I and J of R such that $N.K = (IM).(JM) = IJM = 0$, so $IJ = 0$, and hence either $I = 0$ or $J = 0$. Thus, either $N = IM = 0$ or $K = JM = 0$. Conversely, suppose that $rs = 0$ with $r \neq 0$ for some $r,s \in R$. Then $A = (Rr)M$ and $B = (Rs)M$ are submodules of M with $A.B = 0$. By hypothesis, $B = 0$, and hence $s = 0$, as required. □

3. THE RADICAL OF A SUBMODULE

We believe lemma 3.1 is known, but we do not know an appropriate reference, so we include a proof.

Lemma 3.1 Let R be a commutative ring. If every proper ideal of R is primary, then R is a local ring.

Proof. First, we show that if P and I are ideals of R with P prime, then either $I \subseteq P$ or $P \subseteq I$. Assume that $I \not\subset P$ and choose $a \in I$-P. Let $b \in P$. Then $ab \in PI$ and PI is primary ideal, so we have either $b \in PI$ or $a^m \in PI$ for some m. But if $a^m \in PI \subseteq P$, then $a \in P$, and this contradicts the way a was chosen. Thus we must have $b \in PI$ and hence $P \subseteq PI$. Therefore, $P = PI \subseteq I$. Now let M be any maximal ideal of R. Then M is comparable to any proper ideal. If M' is any maximal, then M and M' are comparable, so $M = M'$, as required. □

Theorem 3.2 Let R be a commutative ring and $M \neq 0$ be an R-module. If every proper sub-module of M is a primary submodule of M and $M \neq T(M)$ where $T(M) = \{m \in M : rm = 0 \text{ for some } r \in R, r \neq 0\}$, then R is a local ring.

Proof. Let $m \in M - T(M)$, so $(0 : m) = 0$, and hence $Rm \cong R$ as R-modules. Clearly, every proper submodule of the R-module $Rm \cong R$ is a primary submodule, hence R is local by lemma 3.1. □

Proposition 3.3 Let $M \neq 0$ be a multiplication module over a commutative ring R. If every proper submodule of M is a primary submodule of M, then M is cyclic.

Proof. This follows from theorem 3.2 and [2, proposition 4]. □

Contrary to what happens in rings, the radical of a primary submodule is not, in general, a prime submodule (see [9, p. 5038]), but we have the following results:

Proposition 3.4 Let M be a multiplication R-module. Then if N is a primary submodule of M, then $\text{Rad}(M)$ is a prime submodule of M.

Proof. Assume that N is a primary submodule of M. Then $I = (N : M)$ is a primary ideal of R by [10, sec. 2.8 proposition 18]. Since $\text{Rad}(I)$ is a prime ideal of R, it follows from [6, corollary 2.11] that $\text{Rad}(N) = \text{Rad}(I)M$ is prime, as required. □

Theorem 3.5 Let R be a one-dimensional domain and M be a P-secondary R-module. Then for any submodule N of M, $\text{Rad}(N)$ is a prime submodule.

Proof. Consider the ideals $(K : M)$ for any

prime submodule K containing N. These ideals are prime and $N \subseteq K$ implies $(N:M) \subseteq (K:M)$ which in turn implies $P = \text{Rad}(N:M) \subseteq \text{Rad}(K:M) = (K:M)$ for all such K (note that M/N is P-secondary by [4, Theorem 2.4]). For one of these prime submodules K, we generate the chain of ideals $0 \subset \text{Rad}(N:M) \subseteq (K:M)$. Then $\text{Rad}(K:M) = (K:M) = P$ for every prime submodule K containing N since $\dim(R) = 1$. Moreover, $(\text{Rad}(N):M) = (\bigcap_{N \subseteq K}(K:M) = \bigcap_{N \subseteq K}(K:M)) = P$. Let $ra \in \text{Rad}(N)$ for some $r \in R$ and $a \in M\text{-Rad}(N)$. Then there exists a prime submodule $N \subseteq K$ of M such that $a \notin K$, so $r \in P$, as required. □

Corollary 3.6 Let M be a secondary module over a Noetherian domain R. If every prime submodule of M contains only finitely many prime submodules, then for any submodule N of M, $\text{Rad}(N)$ is prime.

Proof. By Theorem 3.5, it is enough to show that $\dim(R) = 1$. There exists a free R-module F with $f: F \to M$ an R-module epimorphism. By [11, Result 1.2], if every prime submodule of M contains only finitely many prime submodules, then every prime submodule of F containing $\text{Ker}(f)$ contains only finitely many prime submodules. Now the assertion follows from [9, Corollary 1.4]. □

Corollary 3.7 If R is a one-dimensional domain and M is a torsion module such that 0 is a prime submodule, then for any submodule N of M, $\text{Rad}(N)$ is a prime submodule.

Proof. By Theorem 3.5, it is enough to show that M is a secondary module. As M is torsion, $(0:M) = P \neq 0$ and since 0 is prime, P is a prime ideal of R. Let $a \in R$. If $a \in P$, then $aM = 0$. If $a \notin P$, then by [9, theorem 3.14] we have $aM = M$, as required. □

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