Properties of Perfect Rectangular Donut

Miss Natthanicha Lamaphisarn
Advisor : Asst. Prof. Dr. Khuanchanok Chaichana
Department of Mathematics, Faculty of Science, Chiang Mai University

Abstract

This independent study extends the concept of donuts with Pythagoras, aiming to investigate the relationships of rectangular donuts in various forms and the connection between perfect rectangular donuts and number theory.

Preliminaries

Definition 1 [1]: Let n be an integer. We say that:

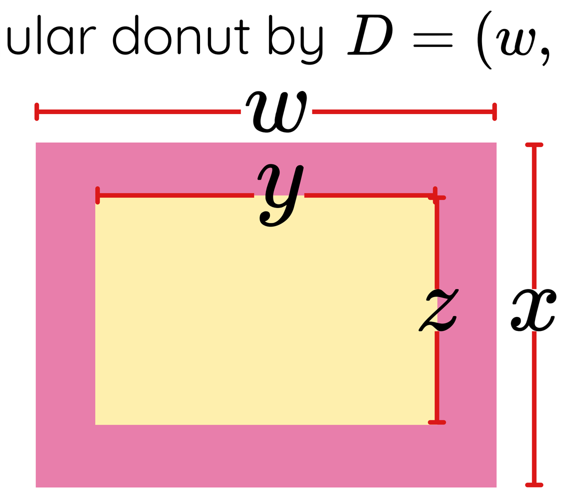
- n is an **even** number if there exists an integer k such that $n = 2k$,
- n is an **odd** number if there exists an integer k such that $n = 2k+1$.

Definition 2 [2]: Let a and b be integers, with at least one of them different from zero. **The greatest common divisor** of a and b, denoted by $gcd\,(a,b)$ is the positive integer d satisfying

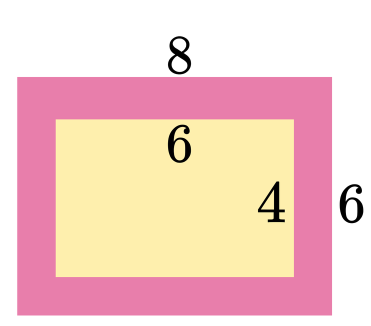
- $d|a$ and $d|b$,
- for all $c \in Z^+$, if $c|a$ and $c|b$, then $c \leqslant d$.

For example: $gcd\,(12,6) = 6$ and $gcd\,(15,8) = 1$.

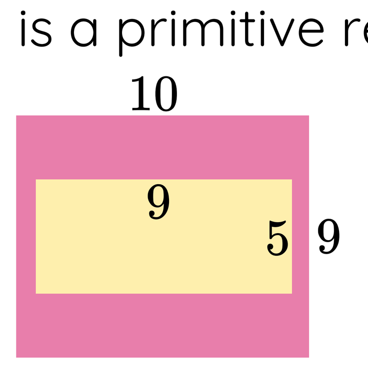
Definition 3 [3]: A rectangular donut is a rectangle with integer sides that contains a smaller rectangle inside it (with sides parallel to the outer rectangle), positioned in the center of the rectangle (representing the hole of the donut). The area of the hole is equal to the area remaining in the original rectangle. Let D be the area of the donut, we represented the following rectangular donut by $D = (w,x,y,z)$.



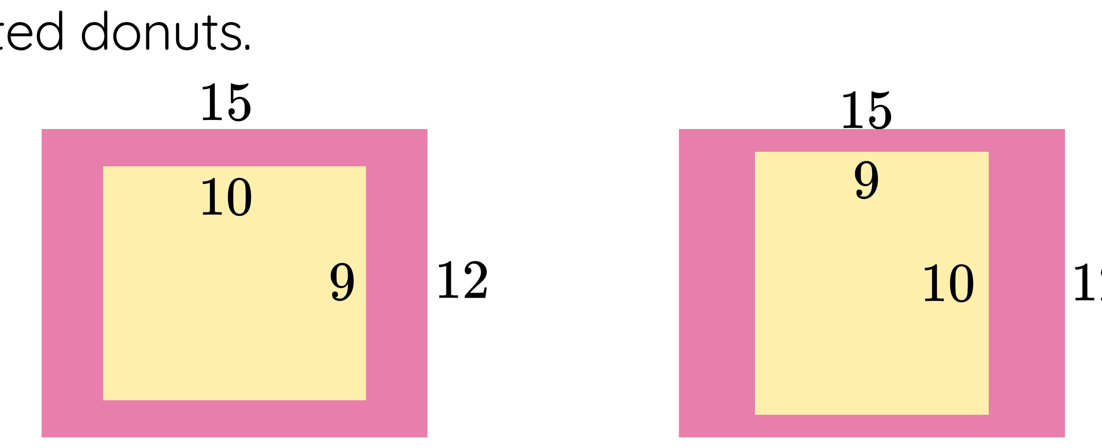
Definition 4 [3]: A perfect rectangular donut (w,x,y,z) has a uniform border with the same integer width on all sides, that is, $w - y = x - z$. For example, $(8,6,6,4)$ is a perfect rectangular donut.



Definition 5 [3]:A primitive rectangular donut (w,x,y,z) is a rectangular donut where $gcd\,(w,y) = 1$ and $gcd\,(x,z) = 1$. For example, $(10,9,9,5)$ is a primitive rectangular donut.



Definition 6 [3]:A twisted donut (w,x,y,z) is a rectangular donut where the length and width of the hole can be swapped or interchanged. For example, $(15,12,10,9)$ and $(15,12,10,9)$ are twisted donuts.



Results

Lemma 1: If (w,x,y,z) is a perfect rectangular donut, then $w - y$ is an even number.

Lemma 2: Let (w,x,y,z) be a perfect rectangular donut. Then, w and x are even numbers if and only if y and z are even numbers.

For example, $(8,6,6,4)$ is a perfect rectangular donut.

Lemma 3: Let (w,x,y,z) be a perfect rectangular donut. Then, w is an even number and x is an odd number if and only if y is an even number and z is an odd number.

For example,$(12,5,10,3)$is a perfect rectangular donut.

Lemma 4: Let (w,x,y,z) be a perfect rectangular donut. Then, w is an odd number and x is an even number if and only if y is an odd number and z is an even number.

For example, $(21,20,15,14)$ is a perfect rectangular donut.

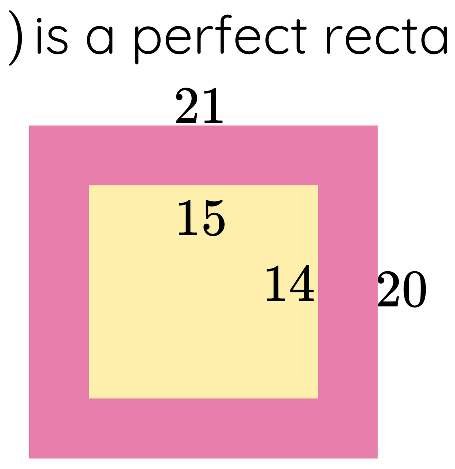
Lemma 5: There is no perfect rectangular donut with all edge lengths being odd.

Theorem 6: For all $k \in Z^+$, (w,x,y,z) is a perfect rectangular donut if and only if $w = 8k$ and $x = 6k$.

For example, $(8,6,6,4)$, $(16,12,12,8)$ and $(24,18,18,12)$.

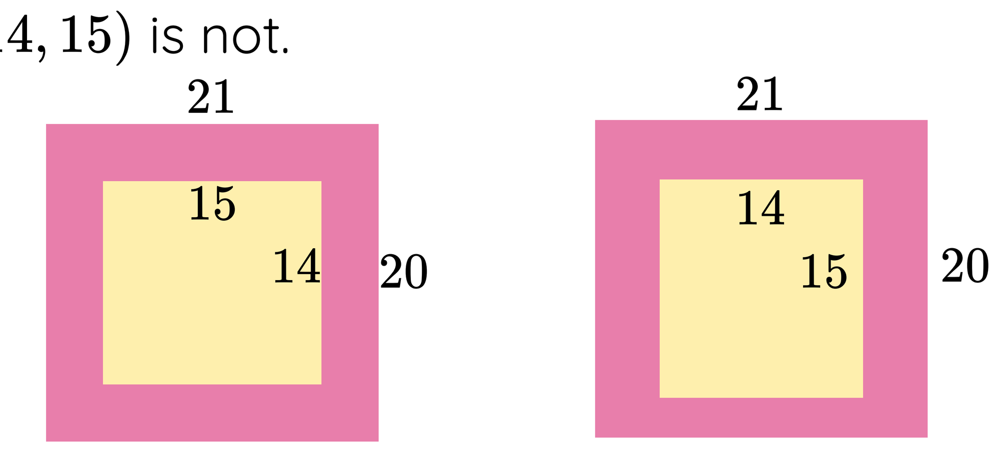
Theorem 7: If (w,x,y,z) is a perfect rectangular donut, then it cannot be a primitive rectangular donut.

For example, $(21,20,15,14)$ is a perfect rectangular donut.



Remark 8: If (w,x,z,y) and (w,x,y,z) are twisted donuts, then there are the following 2 cases:

- **Case 1:** Both twisted donuts are not perfect rectangular donuts. For example, $(15,12,10,9)$, $(15,12,9,10)$ are not perfect rectangular donuts because $15 - 10 \neq 12 - 9$.
- **Case 2:** There is only one configuration that is a perfect rectangular donut. For example, $(21,20,15,14)$ is a perfect rectangular donut, but $(21,20,14,15)$ is not.



References

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