ON THE SOLUTION OF RICCATI EQUATION USING LAPLACE TRANSFORM

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ABSTRACT

In this independent study, we investigate the solution of Riccati equations, which are first-order differential equations in the form of non-linear equations. These equations are challenging to solve. This study will use the Adomian decomposition method combined with the Laplace transform to find an approximate solution of the Riccati equation.

INTRODUCTION

Riccati equation

$$\frac{dy}{dx} = q_1(x) + q_2(x)\,y(x) + q_3(x)\,y^2(x)$$

The Riccati equation is important in various fields because it can simplify complex problems into forms that are easier to solve. In some cases, it allows for transforming complex equations into simpler forms, making it a powerful tool for solving differential equations in nonlinear contexts.

PRELIMINARIES

Nonliear equation an ordinary differential equation

$$y'(t) = f(t, y(t))$$

is called nonlinear. If the function f is nonlinear in the first order. This means that the function $f(t, y(t))$ involves nonlinear expressions in $y(t)$

The initial value problem (IVP) is to find all solutions y of $y(t)$

$$y' = a(t)y + b(t)$$

That satisfy the initial condition $y(t_0) = y_0$

Where a, b are given functions and t_0, y_0 are given constants.

Steps in Adomian decomposition method for non-linear equations

1. Split the non-linear Term.
   $$y'(t) = q(t) + h(t)y(t) + f(t)y^2(t)$$
2. Assume the solution as a Series.
   Let solution $y(t) = \sum_{n=0}^{n} y_n(t)$
3. Decompose the non-linear term using Adomian polynomials.
   Let non-linear term $y^2(t) = \sum_{n=0}^{n} A_n(t)$
4. Apply the decomposition to the equation
5. Solve iteratively for each term
   $$y_n(t) = \int [g(t)]dt$$
6. Check for convergence.

The Laplace transform of a function f defined on $D_f = (0, \infty)$ is

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{b\to\infty}\int_0^b f(t)e^{-st}\,dt$$

Defined for all $s \in D_f \subset \mathbb{R}$ where the integral converges

The inverse Laplace transform denoted L^{-1} of a function $F(s)$ is

$$L^{-1}[F(s)] = f(t)$$

The formulas of the respective Laplace transforms

$f(t)$	$L[f(t)] = F(s)$	$f(t)$	$L[f(t)] = F(s)$
1. 1	$\frac{1}{s}, s > 0$	2. e^{at}	$\frac{1}{s-a}, s > a$
3. $t^n, n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, s > 0$	4. $af + bg$	$aF(s) + bF(s)$
5. $e^{at}f(t)$	$F(s-a)$	6. $f'(t)$	$sF(s) - f(0)$
7. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$	8. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$

MAIN RESULTS

The Laplace transform for the Riccati Equation

Here the Laplace transform is applied to find the solution of

$$\frac{dy}{dt} = 2y(t) - y^2(t) + 1 \quad ; y(0) = 0 \qquad (1)$$

The method consists of applying the Laplace transformation to both sides of (1), where

$$L[y'(t)] = L[2y(t)] - L[y^2(t)] + L[1]. \qquad (2)$$

Applying the formulas of the respective Laplace transforms, we obtain

$$SL[y(t)] - y(0) = L[2y(t)] - L[y^2(t)] + L[1]. \qquad (3)$$

Using the initial condition $y(0) = 0$, we have

$$L[y(t)] = \frac{2}{S}L[y(t)] - \frac{1}{S}L[y^2(t)] + \frac{1}{S}L[1]. \qquad (4)$$

The Laplace transform decomposition technique

$$y(t) = \sum_{n=0}^{n} y_n(t) \quad \text{and} \quad y^2(t) = \sum_{n=0}^{n} A_n(t)$$

Therefore, equation (4) becomes

$$L\left[\sum_{n=0}^{n} y_n(t)\right] = \frac{2}{S}L\left[\sum_{n=0}^{n} y_n(t)\right] - \frac{1}{S}L\left[\sum_{n=0}^{n} y_n(t)\right] - \frac{1}{S}L[1] \qquad (5)$$

Note that the Adomian polynomials $A_n(t)'s$ for the quadratic nonlinearity are written as

$$\begin{aligned} A_0(t) &= y_0^2(t) \\ A_1(t) &= 2y_0(t)\,y_1(t) \\ A_2(t) &= y_1^2(t) + 2y_0(t)\,y_2(t) \\ A_3(t) &= 2y_1(t)\,y_2(t) + 2y_0(t)\,y_3(t) \\ &\vdots \end{aligned} \qquad (6)$$

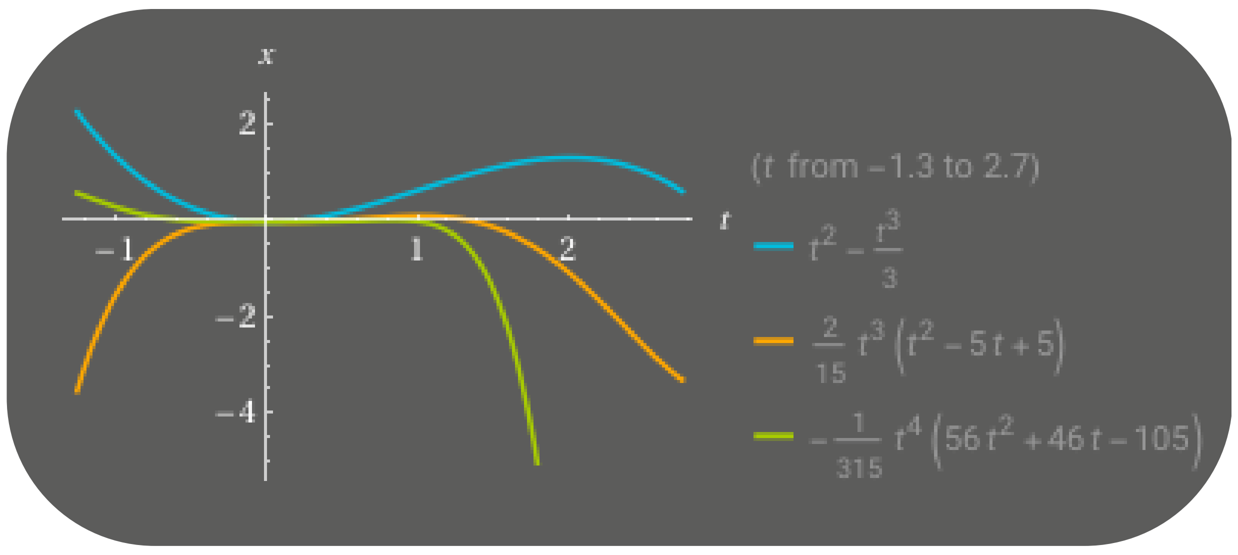
Matching both sides of (5), the following iterative algorithm is obtained.

$$L[y_0] = \frac{1}{S}L[1] = \frac{1}{S^2} \qquad (7)$$

$$L[y_{n+1}(t)] = \frac{2}{S}L[y_n(t)] - \frac{1}{S}L[A_n(t)] \quad ; \quad n > 0 \qquad (8)$$

Substituting $y_n(t)$ and $A_n(t)$ into (8) and applying the inverse Laplace transform, we have

$$y_1(t) = t^2 - \frac{t^3}{3},\ y_2(t) = \frac{2t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{5},\ y_3(t) = \frac{t^4}{3} - \frac{t^5}{5} - \frac{288t^6}{1620} + \frac{357t^5}{6615},$$
$$y_4(t) = \frac{2t^5}{15} - \frac{361t^6}{945} - \frac{608t^7}{2205} - \frac{t^8}{9} + \frac{4t^9}{135}, \ldots$$



CONCLUSION

The study of the Riccati equation and its solutions using the Adomian decomposition method and the Laplace transform method has proven beneficial for solving first-order nonlinear differential equations. These methods help simplify the process of finding solutions and serve as a guideline for solving various other nonlinear differential equations.

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