SOME BASIC PROPERTIES OF HYPERBOLIC *(p, q)*-FIBONACCI OCTONIONS

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Abstract

Octonions are generalization of quaternions. In this work, we are interested in hyperbolic (*p, q*)-Fibonacci octonions, which are structures that generalize octonions and (*p, q*)-Fibonacci numbers. We prove some basic properties of hyperbolic (*p, q*)-Fibonacci octonions, such as the Binet's formulas.

INTRODUCTION

Hamilton invented quaternions in 1843. It can be used to show rotations in three-dimensional space, and have applications in computer graphics, robotics, and aerospace engineering. Quaternions are also used in many areas of mathematics, such as differential geometry and number theory [1].

On the other hand, octonions are generalization of quaternions. In this work, we are interested in hyperbolic (p, q)-Fibonacci octonions, which are structures that generalize (p, q)-Fibonacci numbers.

PRELIMINARIES

Definition 2.1.1 [2] Let p and q be positive integers. Define $F_{p,q,n}$ by

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2} \quad \text{for} \quad n \geq 2$$

with the initial conditions $F_{p,q,0} = 0$ and $F_{p,q,1} = 1$. The sequence $\{F_{p,q,n}\}_{n\geq 0}$ is called the (p, q)-**Fibonacci sequence**. Each term $F_{p,q,n}$ in the (p, q)-Fibonacci sequence is called the n-**th** (p, q)-**Fibonacci number**.

Definition 2.2.1 [2] The **Binet's formulas** for the (p, q)-Fibonacci numbers are defined by

$$F_{p,q,n} = \frac{R_1^n - R_2^n}{R_1 - R_2},$$

where

$$R_1 = \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}$$

are roots of the characteristic equation $R^2 - pR - q = 0$. Then, we have $R_1 + R_2 = p, R_1 - R_2 = \sqrt{p^2 + 4q}$ and $R_1R_2 = -q$.

Definition 2.3.1 [1] A **hyperbolic octonion** ϱ is an element of the form

$$\varrho = \varrho_0 + \varrho_1 i_1 + \varrho_2 i_2 + \varrho_3 i_3 + \varrho_4 \epsilon_4 + \varrho_5 \epsilon_5 + \varrho_6 \epsilon_6 + \varrho_7 \epsilon_7,$$

where $\varrho_0, \varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7$ are real numbers, and $1, i_1, i_2, i_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$ are basis elements satisfying the following properties:

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,$$

$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3,$$

Definition 2.4.1 [3] Let p and q be positive integers, and let $n \geq 0$ be an integer. The **hyperbolic** (p, q)-**Fibonacci quaternion** $HQF_{p,q,n}$ is defined by

$$HQF_{p,q,n} = F_{p,q,n} + F_{p,q,n+1} i_1 + F_{p,q,n+2} i_2 + F_{p,q,n+3} i_3,$$

where $F_{p,q,n}$ is the n-th (p, q)-Fibonacci number. Here $1, i_1, i_2, i_3$ are satisfying the conditions:

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = 1,$$

$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3.$$

OBJECTIVES

1. To study the definition and structure of hyperbolic (p, q)-Fibonacci octonions.
2. To prove some basic properties of hyperbolic (p, q)-Fibonacci octonions, such as the Binet's formulas.

RESULTS

Definition 3.1.1 Let p and q be positive integers, and let $n \geq 0$ be an integer. The **hyperbolic** (p, q)-**Fibonacci octonion** $HOF_{p,q,n}$ is defined by

$$\begin{aligned} HOF_{p,q,n} = {} & F_{p,q,n} + F_{p,q,n+1} i_1 + F_{p,q,n+2} i_2 + F_{p,q,n+3} i_3 \\ & + F_{p,q,n+4} \epsilon_4 + F_{p,q,n+5} \epsilon_5 + F_{p,q,n+6} \epsilon_6 + F_{p,q,n+7} \epsilon_7, \end{aligned}$$

where $F_{p,q,n}$ is the n-th (p, q)-Fibonacci number. Here $1, i_1, i_2, i_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$ are basis elements satisfying the conditions:

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,$$

$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3,$$

$$i_1 \epsilon_4 = \epsilon_5, \quad i_2 \epsilon_4 = \epsilon_6, \quad i_3 \epsilon_4 = \epsilon_7, \quad \epsilon_4^2 = \epsilon_5^2 = \epsilon_6^2 = \epsilon_7^2 = 1.$$

Definition 3.1.3 Let p and q be positive integers, and let $n \geq 0$ be an integer. Then $\overline{HOF_{p,q,n}}$ is defined by

$$\begin{aligned} \overline{HOF_{p,q,n}} = {} & F_{p,q,n} - F_{p,q,n+1} i_1 - F_{p,q,n+2} i_2 - F_{p,q,n+3} i_3 \\ & - F_{p,q,n+4} \epsilon_4 - F_{p,q,n+5} \epsilon_5 - F_{p,q,n+6} \epsilon_6 - F_{p,q,n+7} \epsilon_7, \end{aligned}$$

where $F_{p,q,n}$ is the n-th (p, q)-Fibonacci number. Here $1, i_1, i_2, i_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$ are basis elements satisfying the conditions:

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,$$

$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3,$$

$$i_1 \epsilon_4 = \epsilon_5, \quad i_2 \epsilon_4 = \epsilon_6, \quad i_3 \epsilon_4 = \epsilon_7, \quad \epsilon_4^2 = \epsilon_5^2 = \epsilon_6^2 = \epsilon_7^2 = 1.$$

Theorem 3.2.1 Let p and q be positive integers, and let $n \geq 0$ be an integer. Then the following are true.

1. $HOF_{p,q,n} + \overline{HOF_{p,q,n}} = 2F_{p,q,n}$
2. $HOF_{p,q,n} - \overline{HOF_{p,q,n}} = 2HOF_{p,q,n} - 2F_{p,q,n}$
3. $HOF_{p,q,n}\overline{HOF_{p,q,n}} = \sum_{i=0}^{3} F_{p,q,n+i}^2 - \sum_{i=4}^{7} F_{p,q,n+i}^2$

Theorem 3.3.1 Let p and q be positive integers, and let $n \geq 0$ be an integer. We have

$$HOF_{p,q,n} = \frac{R_1^n \widetilde{R_1} - R_2^n \widetilde{R_2}}{R_1 - R_2},$$

where
$$\begin{aligned} & R_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}, \\ & \widetilde{R_1} = 1 + R_1 i_1 + R_1^2 i_2 + R_1^3 i_3 + R_1^4 \epsilon_4 + R_1^5 \epsilon_5 + R_1^6 \epsilon_6 + R_1^7 \epsilon_7, \\ & \widetilde{R_2} = 1 + R_2 i_1 + R_2^2 i_2 + R_2^3 i_3 + R_2^4 \epsilon_4 + R_2^5 \epsilon_5 + R_2^6 \epsilon_6 + R_2^7 \epsilon_7. \end{aligned}$$

CONCLUSION

In this work, we study the definition and structure of hyperbolic (p, q)-Fibonacci octonions. In addition, we prove some basic properties and Binet's formulas.

REFERENCES

[1] Godase, A. D. (2024). Some new properties of hyperbolic k-Fibonacci and k-Lucas octonions.

[2] Thongkam, B., Butsuwan, K., & Bunya, P. (2020). Some properties of (p, q)-Fibonacci-like and (p, q)-Lucas numbers. Rn, 2(R1), R2.

[3] Yağmur, T. (2020). A note on hyperbolic (p, q)-Fibonacci quaternions. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69(1), 880-890.