



Matrix determinant and subgroups of symmetric group



Pinthong Mueangmun
Assoc. Prof. Dr. Attapol Kaewkhao

Abstract

By studying the method of finding the determinant of a 3×3 matrix, where we repeat the first 2 columns, sum the products of the 3 diagonals from the upper left to the lower right, and subtract the sum of 3 diagonals from the upper right to the lower left, we generalize this idea to $n \times n$ matrix A . We call the quantity obtained the dihedral, denoted by $Dih(A)$. We then explore some properties of $Dih(A)$, in particular, we find some properties that ensure this quantity satisfies $Dih(A) = \det(A)$.

Introduction

One well-known method for computing the determinant of a matrix is Sarrus' rule, which applies specifically to matrices. This method involves expanding along the first row and summing the products of matrix elements along diagonals. However, Sarrus' rule is limited to matrices and cannot be extended to larger matrices. Sarrus introduced the use of signs in determinant calculations and denoted this with the symbol. This concept inspired the generalization of Sarrus' rule to matrices of any size, leading to the definition of a new variable called Δ . This concept builds upon both Sarrus' rule and the traditional determinant. We formally define Δ as a determinant-like function that can be computed for matrices of any size using sign adjustments and specific properties that establish a connection between Δ and \det . This work explores the theoretical foundation of Δ , examines its key properties, and compares it with the traditional determinant, focusing on the conditions under which $\Delta = \det$.

Let $A = [a_{ij}]_{n \times n}$

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

$$dih(A) = \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

$$Dih(A) = \sum_{\sigma \in D_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

Reference

Dirk Lorenz & Karl-Joachim Wirths (2018) Sarrus Rules for Matrix Determinants and Dihedral Groups, The College Mathematics Journal, 49:5, 333-340, DOI: 10.1080/07468342.2018.1526019

objectives

1. To study the various properties of the determinant of a matrix.
2. To examine the subgroups of the symmetric group and their related properties.
3. To investigate the relationship between the determinant and the subgroups of the symmetric group.
4. To construct a new type of determinant using subgroups of the symmetric group.
5. To analyze the properties of the values obtained in 4. and compare them with the traditional determinant.

Results

Theorem 3.1 Let A be an $n \times n$ lower triangular matrix, then $dih(A) = Dih(A) = \det(A)$.

example $A = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ -2 & 4 & 2 & -1 \end{bmatrix}_{4 \times 4}$

$$dih(A) = Dih(A) = \det(A) = -24$$

Theorem 3.2 Let A be an $n \times n$ upper triangular matrix, then $dih(A) = Dih(A) = \det(A)$.

example $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}_{4 \times 4}$

$$dih(A) = Dih(A) = \det(A) = 126$$

Theorem 3.3 Let A be $n \times n$ dimensions. Describe $Dih(A) = Dih(A^T)$

example $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 2 & 8 & 1 & 2 \\ 3 & 9 & 2 & 4 \\ 4 & 0 & 3 & 6 \end{bmatrix}_{4 \times 4}$, $A^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 8 & 9 & 0 \\ 6 & 1 & 3 & 5 \\ 7 & 2 & 4 & 6 \end{bmatrix}_{4 \times 4}$

$$Dih(A) = Dih(A^T) = -978$$

Theorem 3.4 Let A be a nearly upper triangular matrix 4×4 i.e., $a_{21} = 0, a_{23} = 0, a_{31} = 0, a_{41} = 0, a_{42} = 0, a_{43} = 0$ then $dih(A) = Dih(A) = \det(A)$.

example $A = \begin{bmatrix} 9 & 4 & 60 & 1 \\ 0 & 8 & 0 & 6 \\ 0 & 4 & 4 & 55 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{4 \times 4}$

$$dih(A) = Dih(A) = \det(A) = 864$$

Theorem 3.4 Let A be a nearly lower triangular matrix i.e., $a_{21} = 0, a_{23} = 0, a_{31} = 0, a_{41} = 0, a_{42} = 0, a_{43} = 0$ then $dih(A) = Dih(A) = \det(A)$.

example $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 2 & 7 & 0 \\ 7 & 0 & 3 & 0 \\ 7 & 7 & 7 & 4 \end{bmatrix}_{4 \times 4}$

$$dih(A) = Dih(A) = \det(A) = 24$$

Conclusion

In this study, we examined the properties of the determinant of a matrix and extended the concept by defining a new form of determinant using subgroups of the symmetric group. This approach differs from traditional determinant calculations. We proved key theorems that establish the relationship between the newly defined determinant and the classical determinant. Furthermore, for lower triangular matrices, upper triangular matrices, almost lower triangular matrices, and almost upper triangular matrices, we found that $dih(A) = Dih(A) = \det(A)$. This new approach proves to be particularly useful when computing determinants of matrices with dimensions larger than 3×3 , as $Dih(A)$ provides a more efficient way to obtain the result.