Construction of a regular m-gon inscribed in a regular n-gon

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Abstract

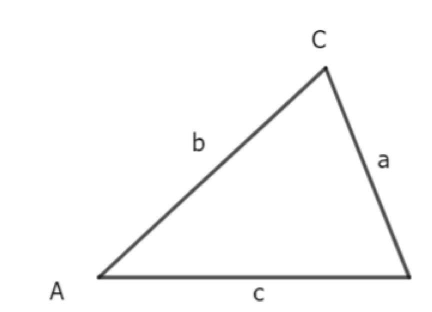
A regular m-gon is a geometric shape with m sides and m interior angles, where all sides have equal length and all interior angles are equal. In this independent study, we consider methods of inscribing a regular m-gon within a regular n-gon, based on the inscribing conditions outlined in the article "Inscribing a regular m-gon in a regular n-gon" by S. J. Dilworth and S. R. Mane, published in the Journal of Geometry (2010). The possible cases of inscription are categorized into four scenarios: (1) $m = 3$. (2) $m = 4$. (3) $m \geq 5$ and m divides n $(m|n)$. (4) $m \geq 6$, m is even, and n is an odd multiple of $m/2$. We will examine the placement of the inscribed polygon in each case to determine the largest possible regular m-gon that can still be inscribed within a regular n-gon.

Preliminaries

We consider methods of inscribing a regular m-gon within a regular n-gon, based on the inscribing conditions outlined in the article "Inscribing a regular m-gon in a regular n-gon" by S. J. Dilworth and S. R. Mane, published in the Journal of Geometry (2010).

Law of sine

Let triangle ABC have sides AB = c, AC = b, and BC = a.
It can be concluded that the ratio of the length of side a to the sine of angle A is equal to the ratio of the length of side b to the sine of angle B, and it is also equal to the ratio of the length of side c to the sine of angle C.

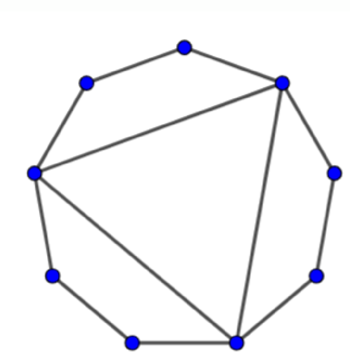


Results

The construction of the largest regular m-gon inscribed in a regular n-gon, based on the distance between the vertices of the regular m-gon and the corresponding vertices of the regular n-gon on the same side, yields the following results:

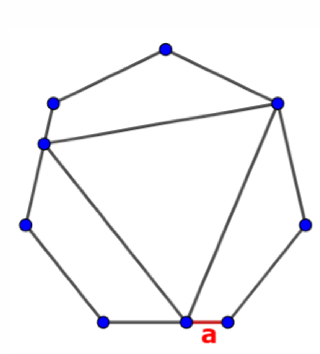
case m=3 is divided into two cases as follows:

1. m divides n
The distance a is: 0



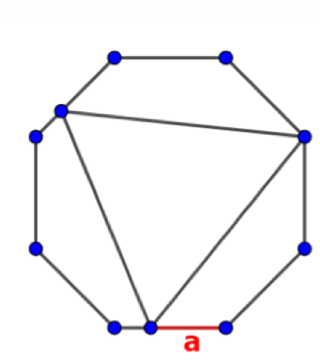
2. m divides n with a remainder of 1

The distance a is: $\dfrac{(x\sin(\frac{(n-2)\pi}{2n})(\sin(\frac{(2n-2)\pi}{3n}))(\sin\frac{\pi}{3n})}{(\sin\frac{2\pi}{n})(\sin\frac{(n+2)\pi}{6n})\sin\frac{(n+1)\pi}{3n}}$



3. m divides n with a remainder of 2

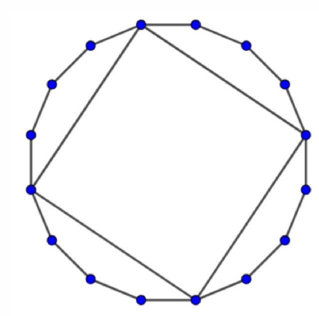
The distance a is: $\dfrac{x\sin(\frac{(n-2)\pi}{2n})(\sin(\frac{(2n-4)\pi)}{3n})(\sin\frac{2\pi}{3n})}{(\sin\frac{2\pi}{n})(\sin\frac{(2n+4)\pi}{6n})(\sin(\frac{(n+1)\pi}{3n}))}$



case m=4 is divided into two cases as follows:

Find the values of a and b from the following system of linear equations with two equations and two variables.

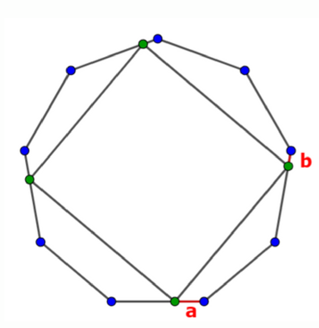
1. m divides n
The distance a and b are: 0



2. m divides n with a remainder of 1

- $\frac{n-1}{4}$ is even **a and b must be consistent with these two equations.:**

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2(x-b)\sin(\frac{(n-3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-1)\pi}{2n})}{\sin(\frac{(n+1)\pi}{4n})\sin(\frac{2\pi}{n}))}\sin(\frac{(n-3)\pi}{4n})$$

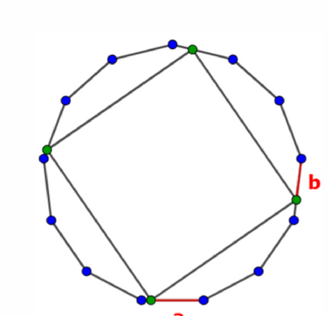


and

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n}))}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2a\sin(\frac{(n+1)\pi}{4n}) + \frac{x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}$$

- $\frac{n-1}{4}$ is odd **a and b must be consistent with these two equations.:**

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2(x-b)\sin(\frac{(n+1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}$$

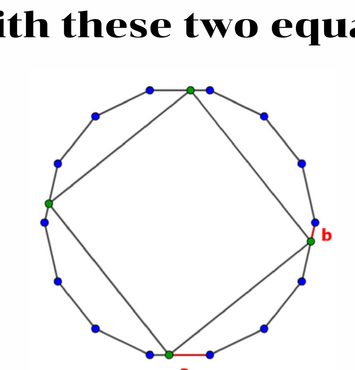


and

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2a\sin(\frac{(n-3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-1)\pi}{2n})}{\sin(\frac{(n+1)\pi}{4n})\sin(\frac{2\pi}{n})}$$

3. m divides n with a remainder of 2
a and b must be consistent with these two equations.:

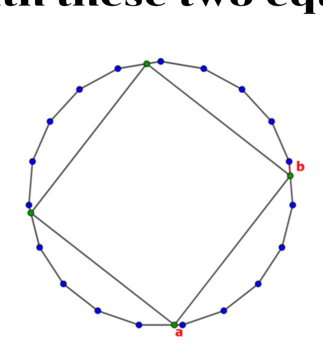
$$2a\sin(\tfrac{(n-2)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-2)\pi}{2n})}{\sin(\frac{(n+2)\pi}{4n})\sin(\frac{2\pi}{n})}$$
$$= 2(x-a)\sin(\frac{(n+2)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-6)\pi}{2n})}{\sin(\frac{(n+6)\pi}{4n})\sin(\frac{2\pi}{n})}$$



4. m divides n with a remainder of 3

- $\frac{n-3}{4}$ is even **a and b must be consistent with these two equations.:**

$$(x-a)\sin(\tfrac{(n-3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-3)\pi}{2n})}{\sin(\frac{(n+3)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+1)\pi}{4n})$$
$$= 2(x-b)\sin(\frac{(n-1)\pi}{4n}) + \frac{(\frac{x\sin(\frac{(n-2)\pi}{2n})}{\sin(\frac{2\pi}{n})})\sin(\frac{(n-3)\pi}{2n})}{\sin(\frac{(n+3)\pi}{4n})}$$

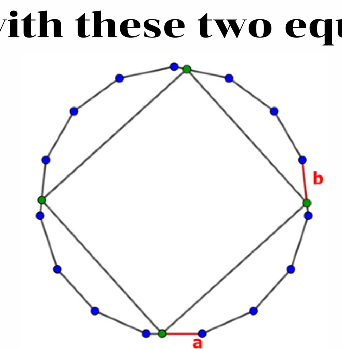


and

$$(x-a)\sin(\tfrac{(n-3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-3)\pi}{2n})}{\sin(\frac{(n+3)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+1)\pi}{4n})$$
$$= 2a\sin(\frac{(n+3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-7)\pi}{2n})}{\sin(\frac{(n+7)\pi}{4n})\sin(\frac{2\pi}{n})}$$

- $\frac{n-3}{4}$ is odd **a and b must be consistent with these two equations.:**

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2(x-b)\sin(\frac{(n+3)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-7)\pi}{2n})}{\sin(\frac{(n+7)\pi}{4n})\sin(\frac{2\pi}{n})}$$



and

$$(x-a)\sin(\tfrac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-5)\pi}{2n})}{\sin(\frac{(n+5)\pi}{4n})\sin(\frac{2\pi}{n})}(\sin(\frac{(n-1)\pi}{2n})) + b\sin(\frac{(n+3)\pi}{4n})$$
$$= 2a\sin(\frac{(n-1)\pi}{4n}) + \frac{(x\sin(\frac{(n-2)\pi}{2n})\sin(\frac{(n-3)\pi}{2n})}{\sin(\frac{(n+3)\pi}{4n})\sin(\frac{2\pi}{n})}$$

Methodology

Given that the side length of an n-gon is x, by using the congruence of triangles, the law of Sine, and the fact that the embedded triangle and square are regular, we can determine the distance between points.

Conclusion

We can construct the largest regular m-gon inscribed in a regular n-gon, based on the distance between the vertices of the regular m-gon and the corresponding vertices of the regular n-gon on the same side.