Application of Haar Wavelet Method for Solving Differential Equations

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1 Abstract

Differential equations are fundamental for analyzing dynamic systems and are widely used in many fields. However, obtaining analytical solutions is often difficult or even impossible. Therefore, numerical methods are essential for solving these equations. This study investigates the fundamental properties of Haar wavelets and **applies the Haar wavelet method to solve differential equations.** The SIR model, a system of nonlinear ordinary differential equations used to study the dynamics of infectious diseases, serves as a case study. This research will approximate solutions to the SIR model under various initial conditions and epidemic scenarios, including the initial, peak, and recovery phases of an outbreak, to evaluate the effectiveness of the Haar wavelet method.

2 Haar Wavelet

The Haar wavelet was first introduced in 1910 by Alfred Haar. It is a simple and effective orthogonal wavelet. The Haar wavelet is defined as a step function on the interval $[0, 1]$.

The scaling function is

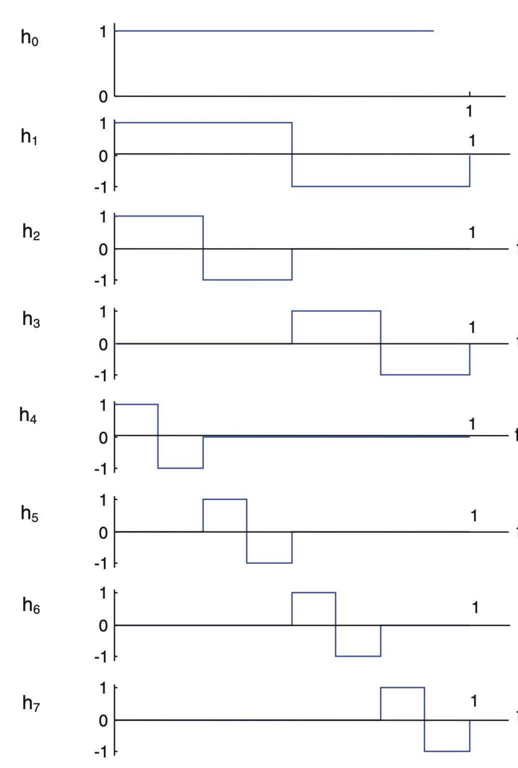
$$h_0(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The mother wavelet is

$$h_1(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The general form of the Haar wavelet is

$$h_n(t) = h_1\left(2^j t - k\right);\ n = 2^j + k,\ j \ge 0,\ 0 \le k < 2^j.$$



One of the important properties of Haar wavelets is that they are an orthogonal basis on $L^2[0,1]$, that is any function $y(t)$ in $L^2[0,1]$ can be represented as a series of Haar wavelets.

$$y(t) = a_0 h_0(t) + a_1 h_1(t) + \cdots = \sum_{n=0}^{\infty} a_n h_n(t) \approx \sum_{n=0}^{m-1} a_n h_n(t) = a^T H_m(t)$$

where $a^T = [a_0\ a_1\ a_2\ \cdots\ a_{m-1}]$ is the vector that contains all of coefficients

and $H_m(t) = \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ \vdots \\ h_{m-1}(t) \end{bmatrix}$ is the Haar matrix of order m

operational matrix computed by $P_m = \left[\int_0^t H_m(u)\, du\right] H_m^{-1}(t)$

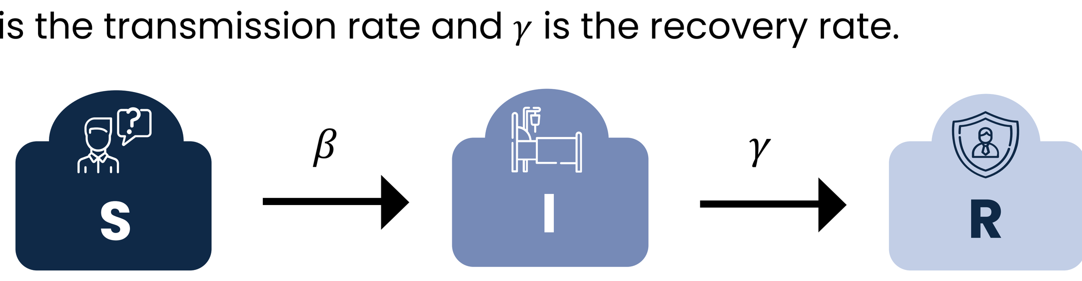
3 SIR Model

The SIR model studies the spread of infectious disease by dividing people into three groups.

1. Susceptible (S): People who are not yet infected and are at risk of infection.
2. Infectious (I): People who are currently infected and can transmit the disease.
3. Recovered (R): People who have recovered from the disease and are immune.

$$\frac{dS}{dt} = -\beta S I, \quad \frac{dI}{dt} = \beta S I - \gamma I, \quad \frac{dR}{dt} = \gamma I$$

Where β is the transmission rate and γ is the recovery rate.



4 Method

The SIR model will be solved under the initial conditions $S(0) = s_0$, $I(0) = i_0$, and $R(0) = r_0$.

Assume that

$$\frac{dS}{dt} \approx \sum_{n=0}^{m-1} a_n h_n(t) = a^T H_m(t) \qquad (4)$$

$$\frac{dI}{dt} \approx \sum_{n=0}^{m-1} b_n h_n(t) = b^T H_m(t) \qquad (5)$$

$$\frac{dR}{dt} \approx \sum_{n=0}^{m-1} c_n h_n(t) = c^T H_m(t). \qquad (6)$$

By integrating (4), (5), and (6) we have

$$S(t) \approx a^T P_m H_m(t) + s_0 \qquad (7)$$

$$I(t) \approx b^T P_m H_m(t) + i_0 \qquad (8)$$

$$R(t) \approx c^T P_m H_m(t) + r_0. \qquad (9)$$

Substitute equations (4)-(9) into the SIR equations, which results in a system of equations. This system will be solved using MATLAB to determine a^T, b^T, and c^T. Finally, S(t), I(t) and R(t) are obtained as desired.

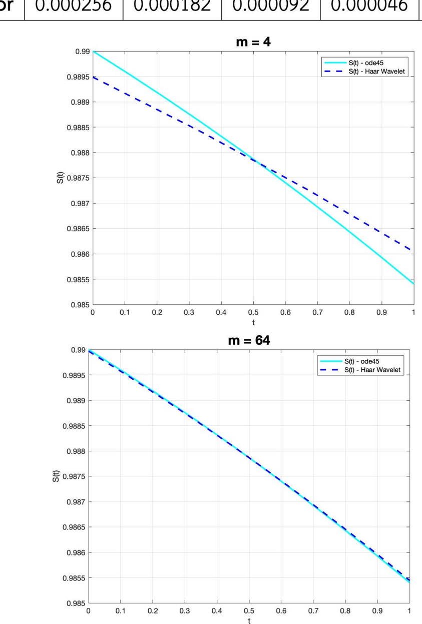
5 Results

We simulate three epidemic stages and compare solutions from the Haar wavelet method and MATLAB's ode45 function, with solid lines for ode45 and dashed lines for Haar wavelet.

Case 1: Early Stage

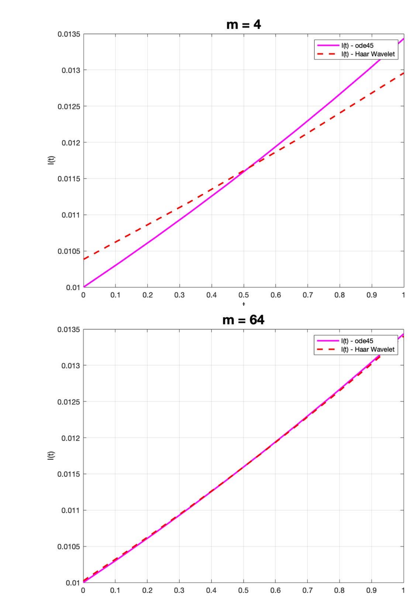
S(t)

m	4	8	16	32	64
Error	0.000256	0.000182	0.000092	0.000046	0.000023



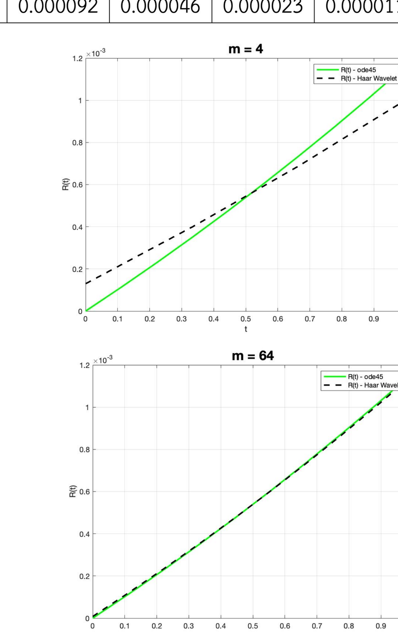
I(t)

m	4	8	16	32	64
Error	0.000271	0.000136	0.000068	0.000034	0.000016



R(t)

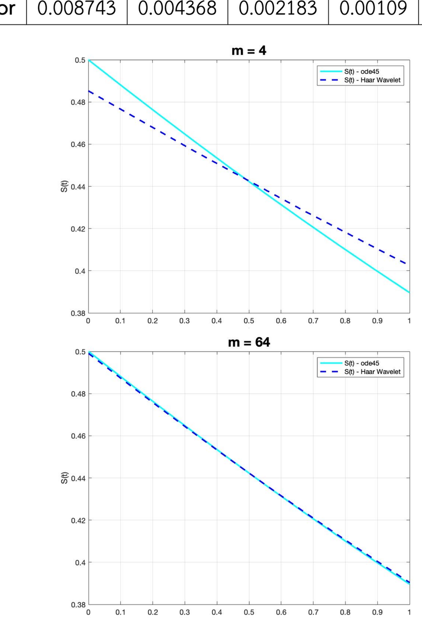
m	4	8	16	32	64
Error	0.000092	0.000046	0.000023	0.000011	0.000005



Case 2: Epidemic peak

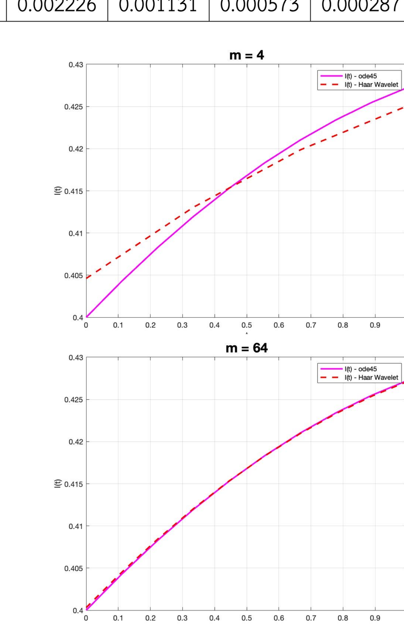
S(t)

m	4	8	16	32	64
Error	0.008743	0.004368	0.002183	0.00109	0.000546



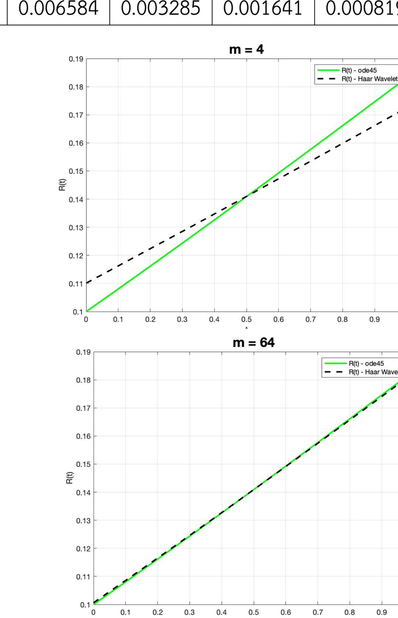
I(t)

m	4	8	16	32	64
Error	0.002226	0.001131	0.000573	0.000287	0.000144



R(t)

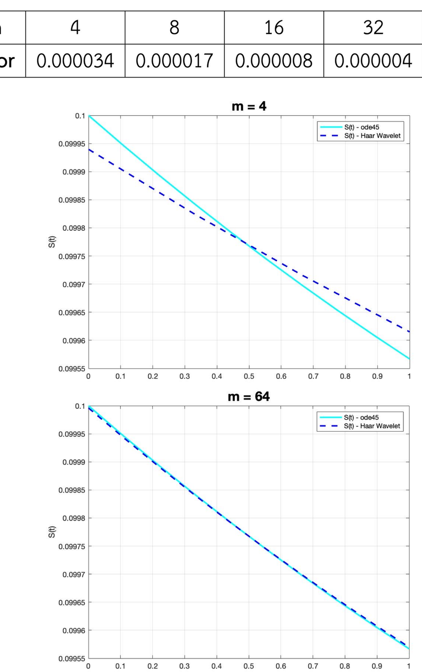
m	4	8	16	32	64
Error	0.006584	0.003285	0.001641	0.000819	0.000410



Case 3: Recovery Stage

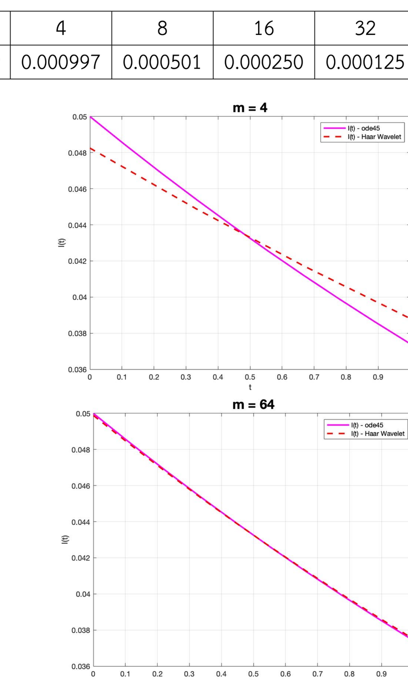
S(t)

m	4	8	16	32	64
Error	0.000034	0.000017	0.000008	0.000004	0.000002



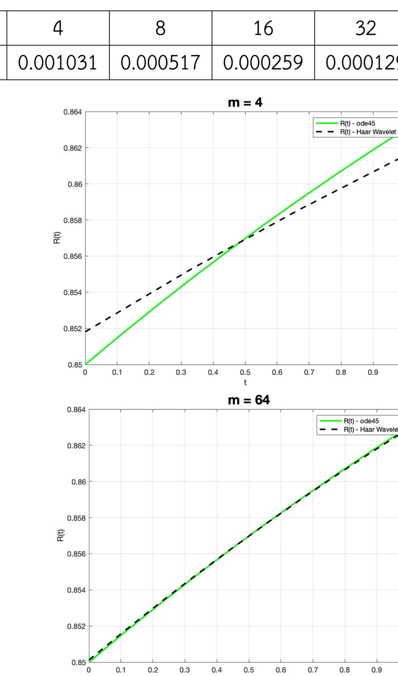
I(t)

m	4	8	16	32	64
Error	0.000997	0.000501	0.000250	0.000125	0.000062



R(t)

m	4	8	16	32	64
Error	0.001031	0.000517	0.000259	0.000129	0.000066



6 Conclusion

This study demonstrates that the Haar wavelet method is an effective tool for solving the differential equations of the SIR model. The method provides results that closely approximate the standard method ode45 and are effective across different stages of an epidemic. Furthermore, as we increase the order of the Haar matrix, or the value of m, the accuracy of the results significantly improves, with the error value decreasing by approximately half with each step of increasing m. However, overly increasing m comes at the cost of higher computational and time demands. Therefore, in practical applications, we should select an appropriate m to balance desired accuracy with available resources.

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