Solving Systems of Nonlinear Equations Using Matrix Equations

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Abstract

In this independent study, we solved certain types of systems of nonlinear equations using knowledge of matrix equations. The method we studied differs from conventional approaches to solving nonlinear systems. Additionally, we derived formulas for the solutions of some types of nonlinear systems.

Introduction

A nonlinear system of equations is a system of equations with multiple variables, where at least one equation is nonlinear. Solving a nonlinear system of equations can be done using various methods, such as the substitution method, which is used when one variable can be expressed in terms of another. In this study, we focus on solving nonlinear systems of equations in the following forms.

$$xy = A_1 y + B_1$$

$$xy = A_2 x + B_2$$

Typically, there are various methods for solving linear systems of equations, some of which can be quite complex. Therefore, we are interested in exploring concepts and finding methods to solve nonlinear systems of equations that simplify the process. The method we focus on in this study is the matrix-based approach.

Objectives

- To study the solutions of certain types of nonlinear equation systems using knowledge of matrices.
- To examine the hypothesis that solving some forms of nonlinear equation systems can be achieved using matrices.

Results

Consider the following nonlinear equation.

$$xy = A_1 y + B_1 \qquad (3.1)$$

$$xy = A_2 x + B_2 \qquad (3.2)$$

Where $A_1, A_2 \neq 0$

Theorem 3.1 The system of equations (3.1) and (3.2) has a solution if and only if there exists a real number c such that

$$c = \left(\frac{c - B_1}{A_1}\right)\left(\frac{c - B_2}{A_2}\right)$$

Subtract k from both sides of equations (3.1) and (3.2)

$$xy - k = A_1 y + B_1 - k$$

$$xy - k = A_2 x + B_2 - k$$

Written in the form of a matrix equation, and assume that $xy \neq k$

$$(xy - k)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y & \frac{B_1 - k}{A_2} \\ \frac{B_2 - k}{A_1} & x \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(xy - k)}\begin{bmatrix} y & \frac{B_1 - k}{A_2} \\ \frac{B_2 - k}{A_1} & x \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\begin{bmatrix} x & -\left(\frac{B_1 - k}{A_2}\right) \\ -\left(\frac{B_2 - k}{A_1}\right) & y \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Therefore

$$x - \left(\frac{B_1 - k}{A_2}\right) = A_1$$

$$y - \left(\frac{B_2 - k}{A_1}\right) = A_2$$

Theorem 3.2 The system of equations (3.1) and (3.2) has the solution as follows

$$x = A_1 + \left(\frac{B_1 - k}{A_2}\right) \quad \text{and} \quad y = A_2 + \left(\frac{B_2 - k}{A_1}\right)$$

when $k = \left(\frac{B_1 - k}{A_2}\right)\left(\frac{B_2 - k}{A_1}\right) = \frac{(B_1 + B_2 + A_1 A_2) \pm \sqrt{(B_1 + B_2 + A_1 A_2)^2 - 4(B_1 B_2)}}{2}$

It is noted that $k = \left(\frac{B_1 - k}{A_2}\right)\left(\frac{B_2 - k}{A_1}\right)$ corresponds to the condition for the existence of solutions in Theorem 3.1

Theorem 3.3 The characteristics of the solutions of systems of equations (3.1) and (3.2) are as follows

- If $(B_1 + B_2 + A_1 A_2)^2 - 4B_1 B_2 > 0$, then the system of equations (3.1) and (3.2) has two distinct solutions.

$$(x, y) = \left(A_1 + \left(\frac{B_1 - k_1}{A_2}\right),\ A_2 + \left(\frac{B_2 - k_1}{A_1}\right)\right)$$

$$(x, y) = \left(A_1 + \left(\frac{B_1 - k_2}{A_2}\right),\ A_2 + \left(\frac{B_2 - k_2}{A_1}\right)\right)$$

when

$$k_1 = \frac{(B_1 + B_2 + A_1 A_2) + \sqrt{(B_1 + B_2 + A_1 A_2)^2 - 4(B_1 B_2)}}{2}$$

$$k_2 = \frac{(B_1 + B_2 + A_1 A_2) - \sqrt{(B_1 + B_2 + A_1 A_2)^2 - 4(B_1 B_2)}}{2}$$

- If $(B_1 + B_2 + A_1 A_2)^2 - 4B_1 B_2 = 0$, then the system of equations (3.1) and (3.2) has one solution.

$$(x, y) = \left(A_1 + \left(\frac{B_1 - k}{A_2}\right),\ A_2 + \left(\frac{B_2 - k}{A_1}\right)\right)$$

when

$$k = \frac{B_1 + B_2 + A_1 A_2}{2}$$

- If $(B_1 + B_2 + A_1 A_2)^2 - 4B_1 B_2 < 0$, then the system of equations (3.1) and (3.2) has no solution.

Theorem 3.4 $x_1 y_1 = k_2$ and $x_2 y_2 = k_1$

Conclusion

From the study, it can be seen that certain types of nonlinear systems of equations can be solved using matrices. It was found that the solutions to such nonlinear systems can be divided into three cases, depending on the value of k in those equations. Solving nonlinear systems using matrices is an alternative approach that helps in finding solutions to these systems.

Reference

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