# ISOMORPHISM CONDITIONS FOR CAYLEY REGULARITY GRAPHS OF SEMIGROUPS

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## ABSTRACT

The Cayley regularity graph $CR(S)$ of a semigroup $S$ is a digraph whose vertex set is the semigroup $S$, and whose directed edge set consists of all ordered pairs $(x, y) \in S \times S$ such that $y$ is a regular part of $x$, that is, $x = xyx$. In this independent study, we aim to find the necessary and sufficient conditions on semigroups for their Cayley regularity graphs to be isomorphic.

## OBJECTIVES OF INDEPENDENT STUDY

1. To understand the definition of Cayley regularity graphs of semigroups.
2. To determine the conditions for isomorphism between Cayley regularity graphs of semigroups.

## PRELIMINARIES

### Semigroups

**Definition 2.1** A semigroup is an ordered pair $(S, *)$ such that $S$ is a non-empty set, and $*$ is an associative binary operation on S.

**Definition 2.2** Let $S$ be a semigroup with a binary operation $*$. An element $x \in S$ is called a zero element if, for all $y \in S$,

$$x * y = y * x = x.$$

### Digraphs

**Definition 2.5** A digraph, also known as a directed graph, is a graph in which the edges have a specific direction. A digraph $G = (V, A)$ consists of:

1. $V$ is a set of vertices,
2. $A \subseteq V \times V$ is a set of directed edges (or arcs), where each directed edge is an ordered pair $(u, v)$ with $u, v \in V$.

**Definition 2.6** Two digraphs $G = (V, A)$ and $H = (W, B)$ are said to be isomorphic if there exists a bijective function $f : V \to W$ such that $(u, v) \in A \iff (f(u), f(v)) \in B$ that means $f$ preserves directed edges. Such a function $f$ is called a digraph isomorphism from $G$ to $H$. If there exists a diagraph isomorphism from $G$ to $H$, we write $G \cong H$.

### Cayley regularity graph of semigroups

**Definition 2.7** A Cayley regularity graph $CR(S)$ of a semigroup $S$ is a digraph whose vertex set is the semigroup $S$ and directed edge set is the set of all ordered pairs $(x, y)$, where $(x, y) \in S \times S$, such that $y$ is a regular part of $x$, that is, $x = xyx$. For each element $s$ in a semigroup $S$, we define $Reg_S(s) = \{t \in S : s = sts\}$.

**Example 2.7.1**

Let $S_2$ be a semigroup with the operation $*$ defined as follows.

| $*$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 3 | 3 | 3 |
| 2 | 1 | 2 | 3 |
| 3 | 3 | 3 | 3 |

$S_2$

From the multiplication of $S_2$, we obtain :

$111 \neq 1$, $121 \neq 1$, $131 \neq 1$,

$212 \neq 2$, $222 = 2$, $232 \neq 2$ and

$313 = 3$, $323 = 3$, $333 = 3$.

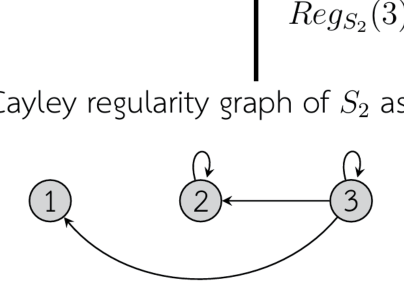
So, we have the regular part of each element as follows :

$Reg_{S_2}(1) = \varnothing$ ,

$Reg_{S_2}(2) = \{2\}$ ,

$Reg_{S_2}(3) = \{1, 2, 3\}$ .

Therefore, we get the Cayley regularity graph of $S_2$ as follows :




$CR(S_2)$

**Definition 2.3** Let $(S_1, *_1)$ and $(S_2, *_2)$ be semigroups. A semigroup isomorphism is a bijective function $f : S_1 \to S_2$ that preserves the operation :

$$f(a *_1 b) = f(a) *_2 f(b), \quad \text{for all } a, b \in S_1.$$

**Remark 2.4** [3] Let $S_1$ and $S_2$ be semigroups and $x$ be a zero element in $S_1$. If $f$ is a semigroup isomorphism from $S_1$ to $S_2$, then $f(x)$ is a zero element in $S_2$.

**Example 2.7.2**

Let $S_1 = \{x, y, z, u\}$ and $S_2 = \{a, b, c, d\}$ be semigroups with operations $*_1$ and $*_2$ defined as follows :

| $*_1$ | $x$ | $y$ | $z$ | $u$ |
|---|---|---|---|---|
| $x$ | $x$ | $x$ | $x$ | $u$ |
| $y$ | $x$ | $x$ | $x$ | $u$ |
| $z$ | $x$ | $x$ | $x$ | $u$ |
| $u$ | $u$ | $u$ | $u$ | $u$ |

$S_1$

| $*_2$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ | $b$ |
| $b$ | $b$ | $a$ | $a$ | $a$ |
| $c$ | $b$ | $a$ | $a$ | $a$ |
| $d$ | $b$ | $a$ | $a$ | $a$ |

$S_2$

Therefore, $S_1$ and $S_2$ are not isomorphic because $S_1$ has a zero element $u$ but $S_2$ does not.

From the multiplication of $S_1$ and $S_2$, the regular parts of each element are :

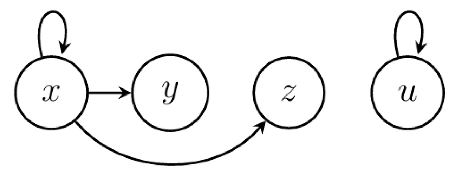
$Reg_{S_1}(x) = \{x, y, z\}$ , $\quad Reg_{S_2}(a) = \{a\}$ ,

$Reg_{S_1}(y) = \varnothing$ , $\quad Reg_{S_2}(b) = \{b, c, d\}$ ,

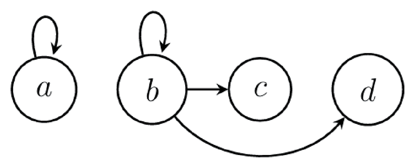
$Reg_{S_1}(z) = \varnothing$, $\quad Reg_{S_2}(c) = \varnothing$,

$Reg_{S_1}(u) = \{u\}$. $\quad Reg_{S_2}(d) = \varnothing$.

From these regular parts, we construct the Cayley regularity graphs $CR(S_1)$ and $CR(S_2)$ shown as below.




$CR(S_1)$




$CR(S_2)$

Let $f : S_1 \to S_2$ such that $f(x) = b$, $f(y) = c$, $f(z) = d$, $f(u) = a$. This implies that $f$ is a digraph isomorphism. Therefore, $CR(S_1)$ and $CR(S_2)$ are isomorphic.

The fact that $CR(S_1)$ and $CR(S_2)$ are isomorphic, while $S_1$ and $S_2$ are not, demonstrates that the Cayley regularity graphs of non-isomorphic semigroups can be isomorphic.

Thus, we aim to determine the conditions on semigroups for their Cayley regularity graphs to be isomorphic.

## MAIN RESULTS

**Theorem 3.1** Let $S_1$ and $S_2$ be semigroups, and $f$ be a digraph isomorphism from $CR(S_1)$ to $CR(S_2)$ such that $f(a) = b$, where $a \in S_1$ and $b \in S_2$. Then, $f(Reg_{S_1}(a)) = Reg_{S_2}(b)$.

**Theorem 3.2** Let $S_1$ and $S_2$ be semigroups, and $f$ be a digraph isomorphism from $CR(S_1)$ to $CR(S_2)$. Then, $a \in Reg_{S_1}(x)$ if and only if $f(a) \in Reg_{S_2}(f(x))$ for all $a, x \in S_1$.

**Theorem 3.3** Let $S_1$ and $S_2$ be semigroups. Then, $CR(S_1) \cong CR(S_2)$ if and only if there exists a bijection $f : S_1 \to S_2$ such that for all $a, x \in S_1$, if $f(a) = b$ and $f(x) = y$, then $x \in Reg_{S_1}(a)$ if and only if $y \in Reg_{S_2}(b)$.

## CONCLUSION

In this work, we study the Cayley regularity graph of a semigroup. First, we obtain two necessary conditions for a function $f : S_1 \to S_2$ to be a digraph isomorphism:

1. $f(\mathrm{Reg}_{S_1}(a)) = \mathrm{Reg}_{S_2}(b)$
2. $a \in \mathrm{Reg}_{S_1}(x)$ if and only if $f(a) \in \mathrm{Reg}_{S_2}(f(x))$.

Finally, we obtain the necessary and sufficient conditions for a function $f : S_1 \to S_2$ to be a digraph isomorphism: $CR(S_1) \cong CR(S_2)$ if and only if there exists a bijection $f : S_1 \to S_2$ such that for all $a, x \in S_1$, if $f(a) = b$ and $f(x) = y$, then $x \in Reg_{S_1}(a)$ if and only if $y \in Reg_{S_2}(b)$.

## REFERENCES

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