# Finding a $b$-matching that Embeds the Maximum Number of Edge Pairs in a Given Set

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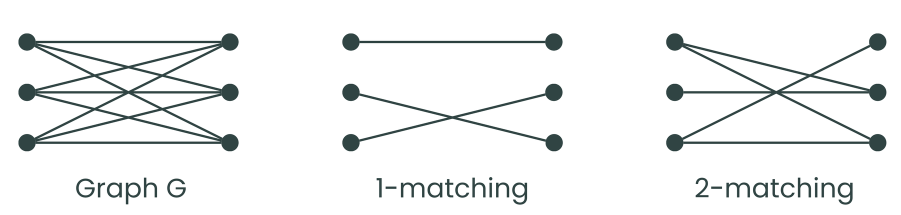
## Abstract

Given a set of edge pairs in a complete bipartite graph, the objective is to find a bipartite $b$-matching that includes the maximum number of these edge pairs. The original problem, known as the maximum edge-pair embedding bipartite matching [1], was demonstrated to be NP-hard and inapproximable by Nguyen et al. in 2021. Building on this, and being inspired by the optimization of reconfigurable networks, we extend the problem in this paper to consider $b$-matchings, with a focus on scenarios where the number of edge pairs per node is bounded. Let $k$ represent the maximum number of edge pairs that can be incident on a single node. We prove that when $k > b$, the problem is NP-hard. For the case when $b = 2$, we provide exact algorithms for $k = 1, 2$. Additionally, for any values of $k$ and $b$, we provide a $\Theta(k)$-approximation algorithm for this problem.

## Preliminaries

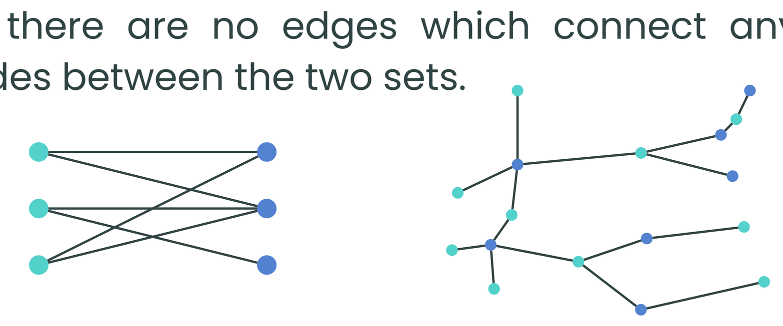
### $b$-matching

A $b$-matching is a subgraph of any graph G such that any node has degree of at most $b$.




### Bipartite Graph

A bipartite graph is a graph such that its set of nodes can be partitioned into two disjoint sets i.e. there are no edges which connect any two nodes between the two sets.



Examples of bipartite graphs, where the two sets of nodes are the set of teal nodes and the set of blue nodes.

### Problem Reduction

To transform a problem P into another problem Q such that it allows us to solve P in terms of Q.

### NP-hardness

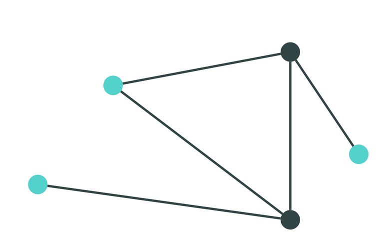
A problem is in the class NP if for all inputs for which the answer is 'yes', this positive answer can be verified efficiently. A problem is NP-hard if all problems in NP can be reduced to it. [2]

Basically, a set of very hard problems.

### Maximum Independent Set problem (MIS)

Given a simple graph $G = (V, E)$, a subset of nodes $S$ of $V$ is an independent set if there are no edges in $E$ between any two nodes of $S$.

The Maximum Independent Set problem asks for a largest independent set of nodes in any given graph. The problem is known to be NP-hard.

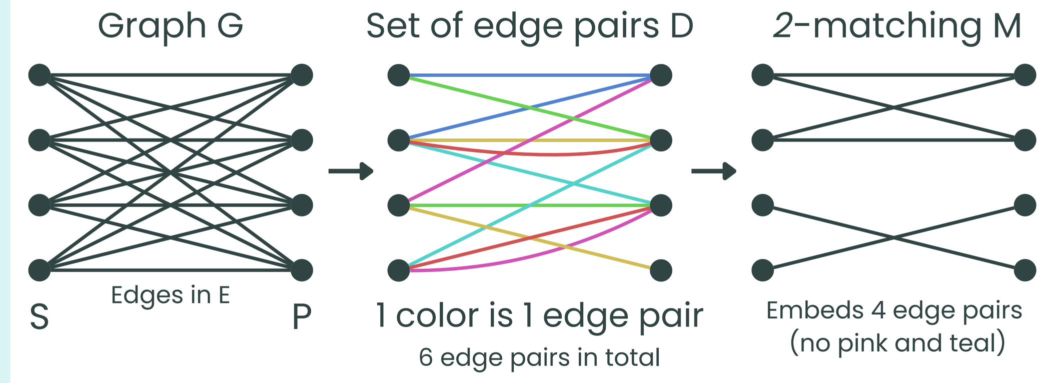


The set of nodes colored in teal is an independent set, and it is a largest independent set of this specific graph.

### Maximum edge-pair embedding bipartite $b$-matching (MEEB$b$M)

Consider two sets of nodes $S = \{s_1, \ldots, s_n\}$ and $P = \{p_1, \ldots, p_m\}$ such that $m \geq n$. Also, consider a complete bipartite graph $G = (S, P, E)$. We are given a set of edge pairs $D \subseteq \{\{e_i, e_j\} : e_i, e_j \in E \text{ and } e_i \neq e_j\}$. We denote the collection of the $b$-matchings by $\mathcal{M}_b$. Our problem, called the *Maximum Edge-pair Embedding Bipartite b-matching (MEEBbM)*, asks for a $b$-matching $M \in \mathcal{M}_b$ that embeds the maximum number of the edge pairs in $D$. The edge pair $\{e_i, e_j\}$ is embedded in $M$ if and only if $M$ includes both $e_i$ and $e_j$. In other words, we want to find

$$\arg\max_{M \in \mathcal{M}_b} |\{d \subseteq M : d \in D\}|.$$



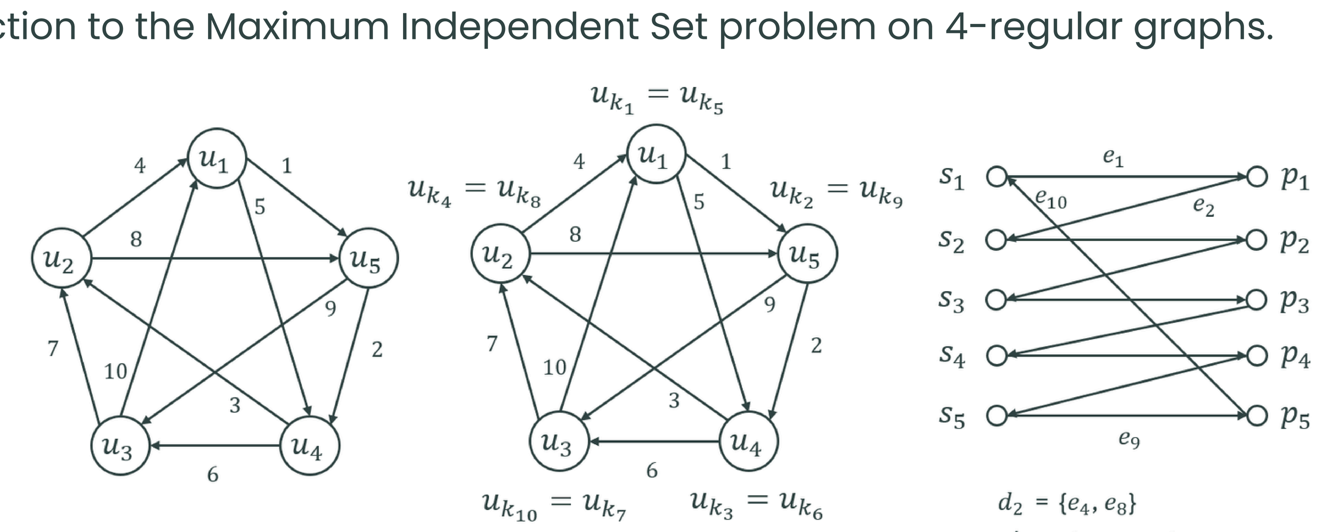

Besides the parameter $b$ (from $b$-matching), we also consider cases such that each node is in less than or equal to $k$ edge pairs. We let $b$ and $k$ be the parameters for this problem.

For example, the above instance has $b = 2$ and $k = 4$.

## Results for MEEB$b$M

### NP-hardness of MEEB$b$M

We have proven that the problem MEEBbM is NP-hard for $k > b$. The proof uses a reduction to the Maximum Independent Set problem on 4-regular graphs.




### Trivial Cases

By the nature of $b$-matchings, if $k$ is less than $(b-1)/2$, then we can embed every edge pair in M, and so these cases are solvable in polynomial time.

### $b = 2$

For k = 1, the problem is trivial, as the solution is to embed every edge pair in D.

For k = 2, the problem can be solved in polynomial time by reducing the problem to a Maximum Independent Set (MIS) problem on bipartite graphs.

### Arbitrary $b$

This approximation algorithm can be used for any $k$ and $b$. It has an approximation ratio of $4k-3$, i.e. $\frac{\text{Objective value of optimal solution}}{\text{Objective value of any approximate solution}} \leq 4k-3$.

**Input:** The set of edge pairs $D$

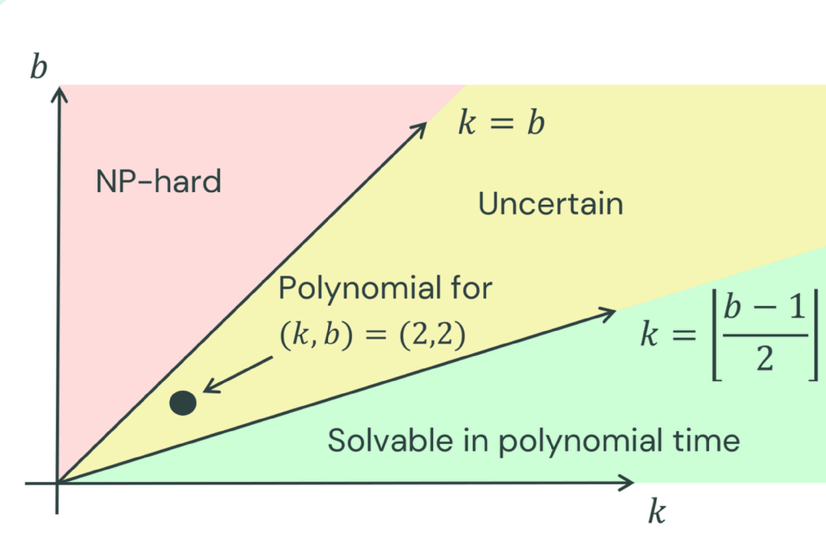
**Output:** (1) A $b$-matching $M$. (2) A set of edge pairs in $D$ contained in $M$, denoted by $S$.

```
M ← ∅, S ← ∅ ;
while there is d = {e_1, e_2} such that d ∉ S and M ∪ d is a b-matching do
    M ← M ∪ d, S ← S ∪ {d}
end
return M, S
```

## Conclusion

In this work, we generalize the maximum edge-pair embedding bipartite matching (MEEBM), a special case of the wireless localization matching problem, to the $b$-matching framework (MEEB$b$M). We demonstrate that:

1. MEEB$b$M is NP-hard for any $k > b$.
2. For the case where $b = 2$, we provide a polynomial time algorithm when $k = 1$ or 2.
3. The problem becomes NP-hard for $k > 2$.
4. There exists a $(4k - 3)$-approximation algorithm for the generalized problem for arbitrary values of $k$ and $b$.




This graph shows the hardness of MEEBbM for different values of $k$ and $b$. The only part of the yellow area we have proven to be solvable is $(k, b) = (2, 2)$.

**Keywords**

Quadratic Assignment, Approximation Algorithm, Computational Complexity, $b$-matching.

**Related literature**

[1] Cam Ly Nguyen, Vorapong Suppakitpaisarn, Athasit Surarerks, and Phanu Vajanopath. On the maximum edge-pair embedding bipartite matching. Theoretical Computer Science, 2021.

[2] Zoltan Mann. The top eight misconceptions about np-hardness. IEEE Computer, 50:72–79, 05 2017.

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