The number of Homomorphisms from the Cartesian product of paths $P_n \times P_2$ to the path P_n

Abstract

Let G and H be graphs. A mapping f from the vertices of G to the vertices of H is known as a homomorphism from G to H if for every pair of adjacent vertices x and y in G, the vertices $f(x)$ and $f(y)$ are adjacent in H. Our work aims to find the cardinalities of the homomorphisms from the cartesian product paths $P_n \times P_2$ to the path P_n ($|Hom(P_n \times P_2, P_n)|$).

Introduction

In 2009, Arworn [1] introduced the algorithm for computing the cardinalities of the endomorphism monoids of finite undirected paths. This number is calculated by the summation of the numbers of shortest paths from point $(0,0)$ to any point (i,j) in a square lattice and an r-ladder square lattice. Later Arworn and Wojtylak [2] gave the number of homomorphisms from an arbitrary path to another arbitrary path. Therefore, we are interested in finding the number of homomorphisms from the Cartesian product of paths $P_n \times P_2$ to the path P_n .

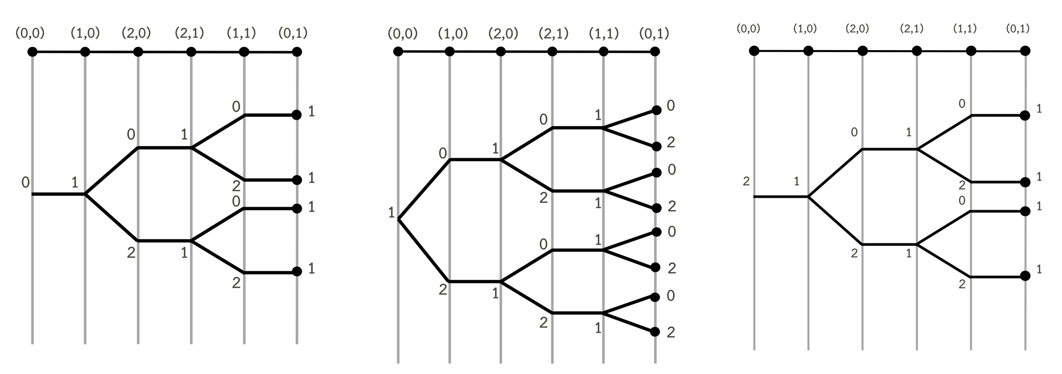
Objectives

- To provide a formula to determine the cardinalities of the homomorphisms from the cartesian product of paths $P_n \times P_2$ to the path P_n .

Method

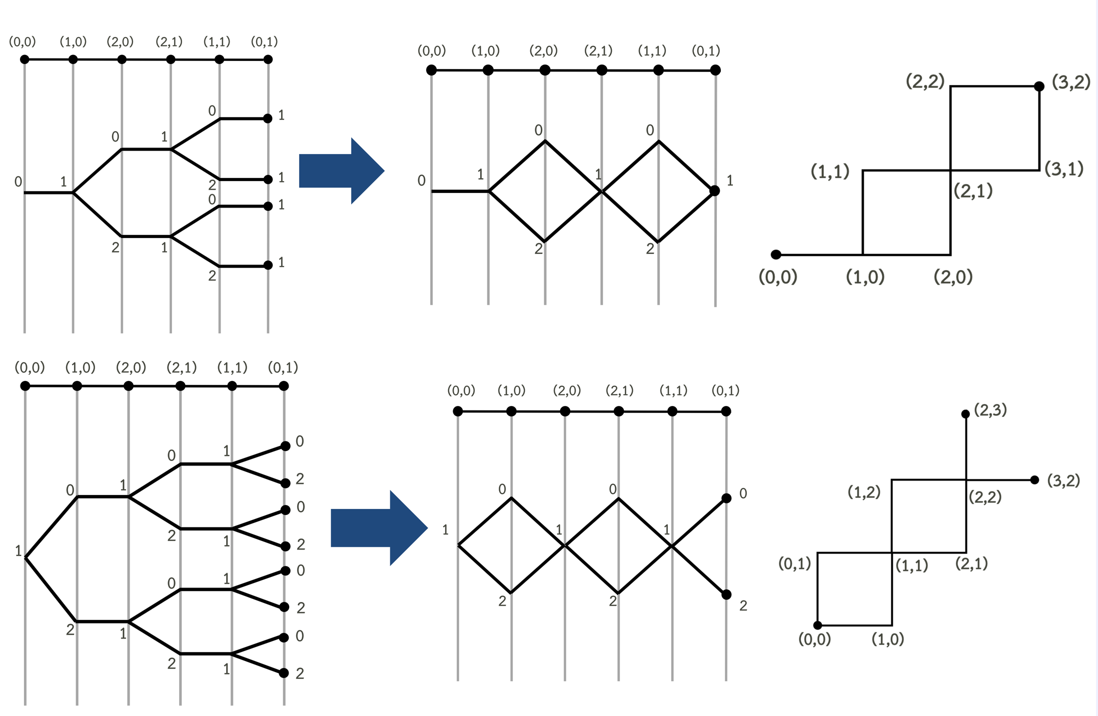
- Consider and create decision tree presentation of all possible homomorphisms.

Example : The homomorphisms from $P_3 \times P_2$ to P_3 which map (0,0) to 0, 1, and 2, respectively ($\mathrm{Hom}^{00}(P_3 \times P_2, r)$ for r=0,1,2).



- Create graphicals and square lattice presentations.

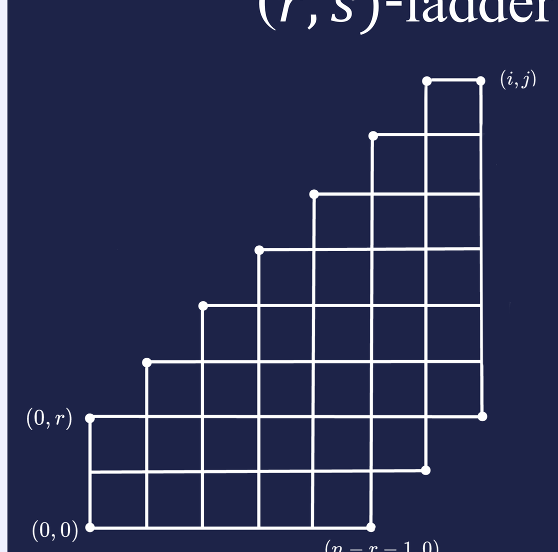
By identifying vertices with identical labels in each vertical line.



The cardinality of $\mathrm{Hom}^{00}(P_3 \times P_2, r)$ corresponds to the number of paths of length 5 from point (0,0) to the point (3,2) or (2,3) for r=0,1,2.

For the general case, the cardinal number of $\mathrm{Hom}^{00}(P_n \times P_2, r)$ corresponds to the number of shortest paths from point (0, 0) to point (n−1, n) or point (0, 0) to point (n, n−1) in an (r,s)-ladder square lattice where $s = n - r - 1$.

(r,s)-ladder square lattice



Let $M_{r,s}(i,j)$ denote the number of shortest paths from the point (0,0) to any point (i, j) in an (r,s)-ladder square lattice.

Let $r, s \geq 0$, $i \geq s+1$ and $j \geq r+1$ be integers. Then on the (r,s) -ladder square lattice, we have

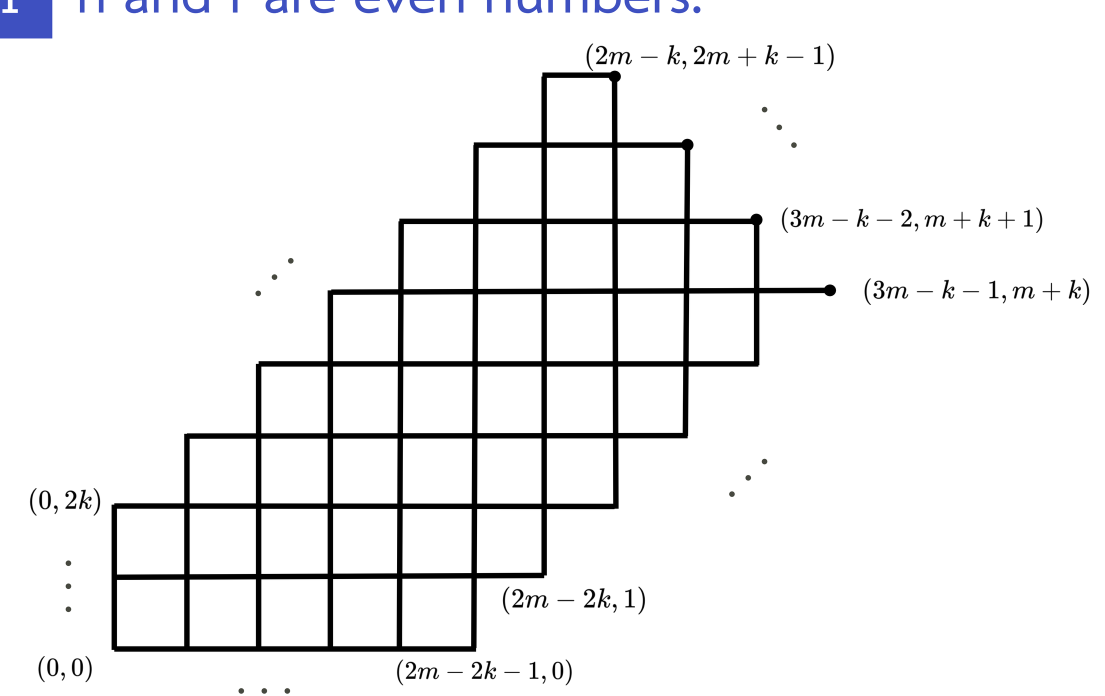
$$M_{r,s}(i,j) = \binom{i+j}{j} - \binom{i+j}{j-r-1} - \binom{i+j}{i-s-1}$$

such that $s = n - r - 1$.

Result

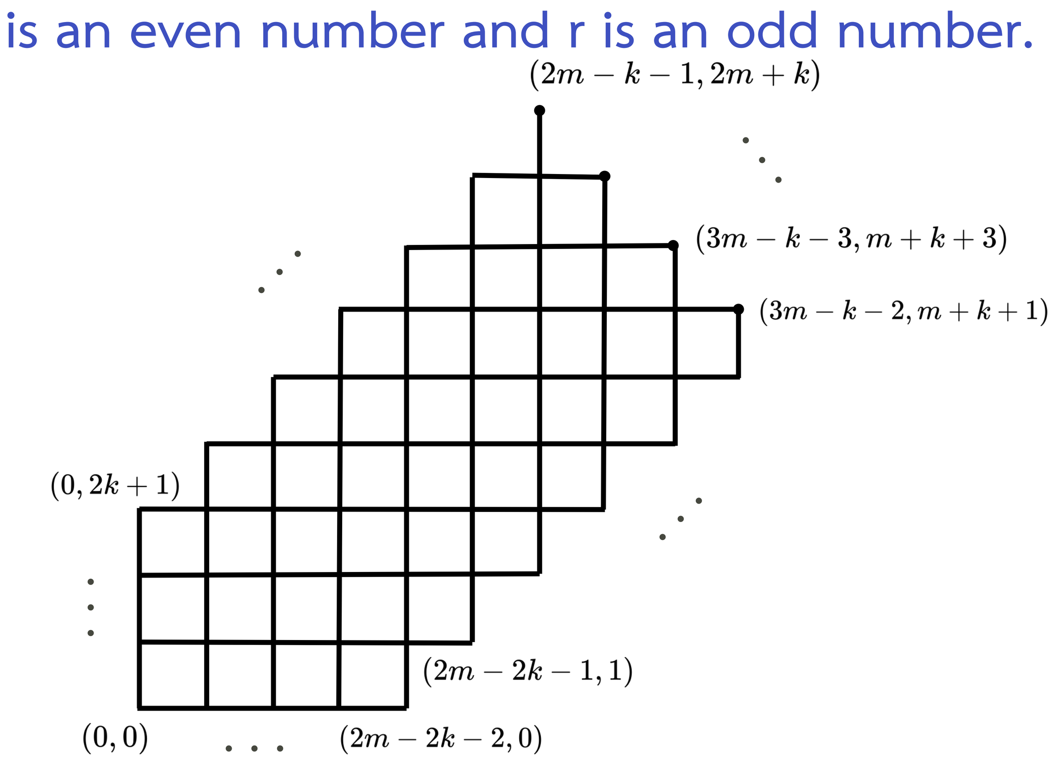
The order of the set of homomorphisms from $P_n \times P_2$ to P_n that map (0,0) to r. Divide the calculation into 4

1 n and r are even numbers.



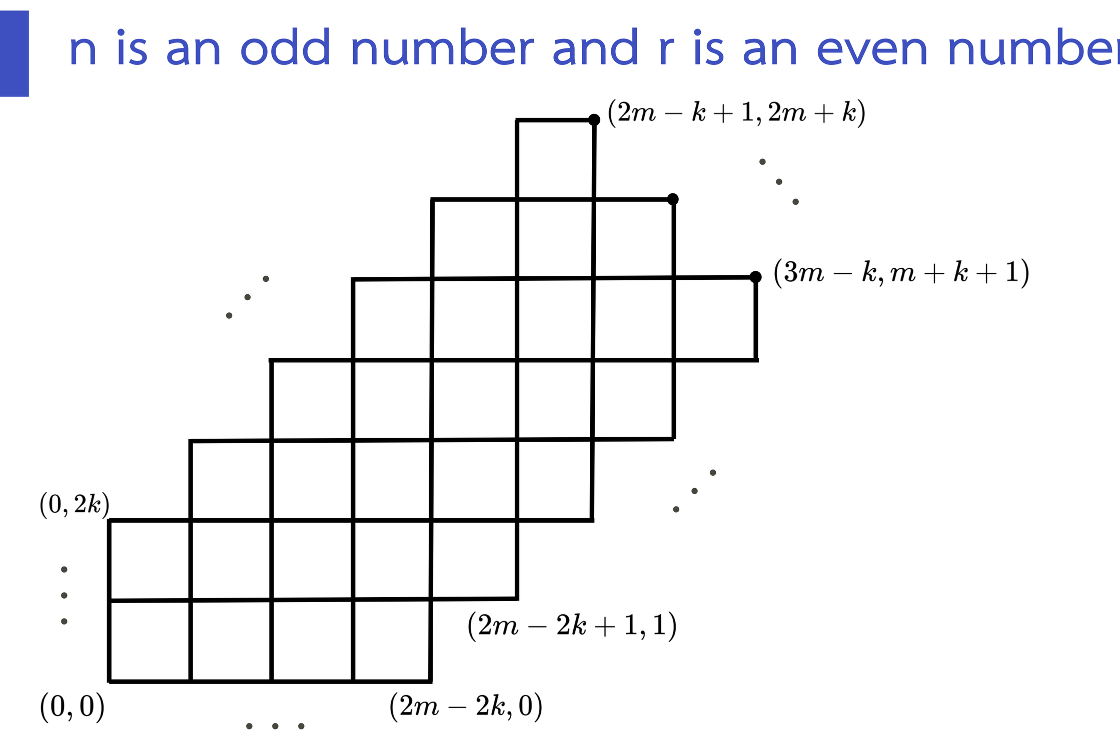
$$|Hom^{00}(P_{2m} \times P_2, 2k)| = M_{2k,2m-2k-1}(2m-1,2m) + M_{2k,2m-2k-1}(2m,2m-1)$$
$$= \left[\binom{4m-1}{2m} - \binom{4m-1}{2m-2k-1} - \binom{4m-1}{2k-1}\right] + \left[\binom{4m-1}{2m-1} - \binom{4m-1}{2m-2k-2} - \binom{4m-1}{2k}\right].$$

2 n is an even number and r is an odd number.



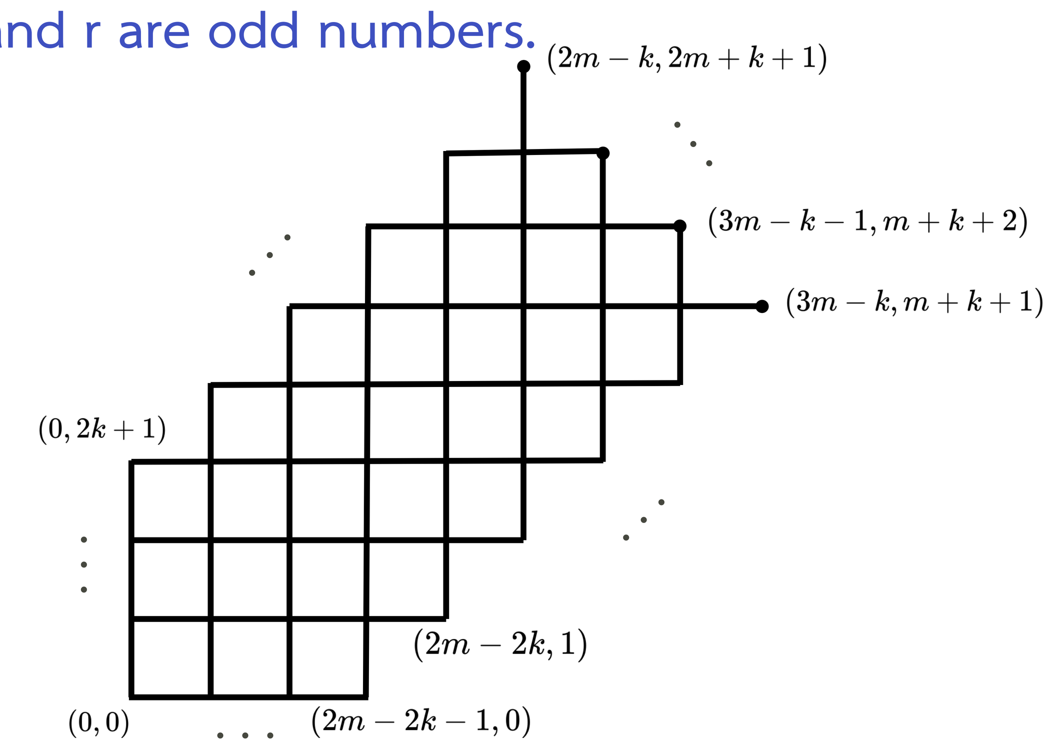
$$|Hom^{00}(P_{2m} \times P_2, 2k+1)| = M_{2k+1,2m-2k-2}(2m-1,2m) + M_{2k+1,2m-2k-2}(2m,2m-1)$$
$$= \left[\binom{4m-1}{2m} - \binom{4m-1}{2m-2k-2} - \binom{4m-1}{2k}\right] + \left[\binom{4m-1}{2m-1} - \binom{4m-1}{2m-2k-3} - \binom{4m-1}{2k+1}\right].$$

3 n is an odd number and r is an even number.



$$|Hom^{00}(P_{2m+1} \times P_2, 2k)| = M_{2k,2m-2k}(2m,2m+1) + M_{2k,2m-2k}(2m+1,2m)$$
$$= \left[\binom{4m+1}{2m+1} - \binom{4m+1}{2m-2k} - \binom{4m+1}{2k-1}\right] + \left[\binom{4m+1}{2m} - \binom{4m+1}{2m-2k-1} - \binom{4m+1}{2k}\right].$$

4 n and r are odd numbers.



$$|Hom^{00}(P_{2m+1} \times P_2, 2k+1)| = M_{2k+1,2m-2k-1}(2m,2m+1) + M_{2k+1,2m-2k-1}(2m+1,2m)$$
$$= \left[\binom{4m+1}{2m+1} - \binom{4m+1}{2m-2k-1} - \binom{4m+1}{2k}\right] + \left[\binom{4m+1}{2m} - \binom{4m+1}{2m-2k-2} - \binom{4m+1}{2k+1}\right].$$

Therefore, we get $|Hom^{00}(P_n \times P_2, r)| = \left[\binom{2n-1}{n} - \binom{2n-1}{n-r-1} - \binom{2n-1}{r-1}\right] + \left[\binom{2n-1}{n-1} - \binom{2n-1}{n-r-2} - \binom{2n-1}{r}\right].$

Conclusion

We determined that the number of homomorphisms from the Cartesian product of paths $P_n \times P_2$ to the path P_n is equal to the summation of the numbers of the shortest paths from point (0, 0) to point (n − 1, n) or point (0, 0) to point (n, n − 1) in (r,s)-ladder square lattices. This can be expressed as

$$|Hom\,(P_n \times P_2, P_n)| = \sum_{r=0}^{n-1} \left[\binom{2n-1}{n} - \binom{2n-1}{n-r-1} - \binom{2n-1}{r-1}\right] + \left[\binom{2n-1}{n-1} - \binom{2n-1}{n-r-2} - \binom{2n-1}{r}\right].$$

References

[1] Srichan Arworn, An algorithm for the numbers of endomorphisms on paths, Journal of Discrete Mathematical Sciences, 2009; 309: page 94-103.

[2] Srichan Arworn and Piotr Wojtylak, An algorithm for the number of path homomorphisms, Journal of Discrete Mathematical Sciences, 2009; 309: page 5569-5573.

Advisor Assoc. Prof. Dr. Sayan Panma. **Author** Pharinda Sonngam 640510490 Pharinda_son@cmu.ac.th