The smallest prime that is not the sum or difference of powers of some two prime numbers

Nattida Sanphor and Assoc. Prof. Dr. Anchalee Khemphet
Department of Mathematics, Faculty of Science, Chiang Mai University

Abstract

Previous research has proven that 53 is the smallest prime number that cannot be written in the form of the sum or difference of powers of 2 and 3. This study extends that scope to investigate the smallest prime number that cannot be represented in the form of the sum or difference of powers of 2 and q for each certain prime number q. Our findings indicate that 19 is the smallest prime number that cannot be written as the sum or difference of powers of 2 and 5. The prime 13 is the smallest prime number that cannot be written as the sum or difference of powers of 2 and 7, nor of 2 and 13. The prime 11 is the smallest prime number that cannot be written as the sum or difference of powers of 2 and 11, nor of 2 and 17.

Introduction

The study of representing integers as either the sum or difference of powers of some two prime numbers is an important topic in number theory, particularly when considering that certain prime numbers cannot be expressed as the sum or difference of powers of two given prime numbers.

Previous research has shown that 53 is the smallest prime number that cannot be written in the form of the sum or difference of powers of 2 and 3. The objective of this research is to extend the scope of that study to a more general case by considering the prime number 2 and prime number q. The aim is to investigate the smallest prime number that cannot be represented as the sum or difference of powers of these two given prime numbers.

This study employs concepts from number theory, particularly the analysis of modular arithmetic, the classification of possible values of powers of the given prime numbers, and the examination of conditions under which such representations are impossible. Consequently, this research generalizes the result from the specific case of 2 and 3 to the broader case of 2 and a certain prime number q.

Objectives

1. To study the characteristics and structure of the smallest prime number that cannot be expressed as the sum or difference of powers of two prime numbers.
2. To extend the idea from the case study of the pair of prime numbers 2 and 3 to other pairs of prime numbers such as 2 and 5, 2 and 7, 2 and 11, 2 and 13, and 2 and 17, in order to determine the smallest prime number that cannot be written in the form of the sum or difference of powers of two prime numbers.

Results

Lemma 1 Let p, q be odd prime numbers and $m, n \in \mathbb{N} \cup \{0\}$. If $p^m \pm q^n$ is a prime number, then $p^m \pm q^n = 2$.

Remark 2 The number 2 is the smallest prime number and the only even prime number.

Lemma 3 Let q be an odd prime number. Then

1. $2^m + q^n = 2$ for some $m, n \in \mathbb{N} \cup \{0\}$,
2. $2^m + q^n = 3$ for some $m, n \in \mathbb{N} \cup \{0\}$,
3. $2^m + q^n = 5$ for some $m, n \in \mathbb{N} \cup \{0\}$,
4. $2^m - q^n = 7$ for some $m, n \in \mathbb{N} \cup \{0\}$.

Remark 4 Let q be an odd prime number. The prime numbers $2, 3, 5$, and 7 can always be written in the form of the sum or difference of 2^m and q^n for some $m, n \in \mathbb{N} \cup \{0\}$.

Results

Lemma 5 Let $k \in \mathbb{N} \cup \{0\}$. Then $2^k \equiv \begin{cases} 2 \pmod 3 & \text{if } k \text{ is odd,} \\ 1 \pmod 3 & \text{if } k \text{ is even.} \end{cases}$

Lemma 6 Let $k \in \mathbb{N} \cup \{0\}$. Then $5^k \equiv \begin{cases} 5 \pmod 8 & \text{if } k \text{ is odd,} \\ 1 \pmod 8 & \text{if } k \text{ is even.} \end{cases}$

Lemma 7 Let $k \in \mathbb{N} \cup \{0\}$. Then $7^k \equiv \begin{cases} -1 \pmod 8 & \text{if } k \text{ is odd,} \\ 1 \pmod 8 & \text{if } k \text{ is even.} \end{cases}$

Lemma 8 Let $k \in \mathbb{N} \cup \{0\}$. Then $11^k \equiv \begin{cases} -1 \pmod 4 & \text{if } k \text{ is odd,} \\ 1 \pmod 4 & \text{if } k \text{ is even.} \end{cases}$

Example 9 The prime numbers $11, 13$, and 17 can be written in the form of the sum or difference of 2^m and 5^n for some $m, n \in \mathbb{N} \cup \{0\}$.

1. Consider $2^m - 5^n = 11$ when $m = 4$ and $n = 1$. Then $2^m - 5^n = 2^4 - 5^1 = 16 - 5 = 11$.
2. Consider $2^m + 5^n = 13$ when $m = 3$ and $n = 1$. Then $2^m + 5^n = 2^3 + 5^1 = 8 + 5 = 13$.
3. Consider $2^m + 5^n = 17$ when $m = 4$ and $n = 0$. Then $2^m + 5^n = 2^4 + 5^0 = 16 + 1 = 17$.

Theorem 10 The smallest prime number that cannot be written in the form of the sum or difference of 2^m and 5^n for all $m, n \in \mathbb{N} \cup \{0\}$ is 19.

Example 11 The prime number 11 can be written in the form of the sum or difference of 2^m and 7^n for some $m, n \in \mathbb{N} \cup \{0\}$.

Consider when $m = 2$ and $n = 1$, we obtain $2^m + 7^n = 2^2 + 7^1 = 4 + 7 = 11$. That is $2^m + 7^n = 11$.

Theorem 12 The smallest prime number that cannot be written in the form of the sum or difference of 2^m and 7^n for all $m, n \in \mathbb{N} \cup \{0\}$ is 13.

Theorem 13 The smallest prime number that cannot be written in the form of the sum or difference of 2^m and 11^n for all $m, n \in \mathbb{N} \cup \{0\}$ is 11.

Example 14 The prime number 11 can be written in the form of the sum or difference of 2^m and 13^n for some $m, n \in \mathbb{N} \cup \{0\}$.

Consider when $m = 1$ and $n = 1$, we obtain $13^n - 2^m = 13^1 - 2^1 = 13 - 2 = 11$. That is $13^n - 2^m = 11$.

Theorem 15 The smallest prime number that cannot be written in the form of the sum or difference of 2^m and 13^n for all $m, n \in \mathbb{N} \cup \{0\}$ is 13.

Theorem 16 The smallest prime number that cannot be written in the form of the sum or difference of 2^m and 17^n for all $m, n \in \mathbb{N} \cup \{0\}$ is 11.

Conclusion

This research investigates the smallest prime number that cannot be expressed as the sum or difference of powers of two given prime numbers. In this study, the two primes considered are 2 and q, where q is either $5, 7, 11, 13$ or 17.

Therefore, this study extends the results from a specific case to a more general setting and reveals the structure of prime numbers that cannot be represented as the sum or difference of powers of two primes. This may provide a direction for further study on general conditions in number theory related to prime numbers.

Reference

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