CONNECTIVITY OF k-DISTANCE PRODUCT GRAPHS

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Abstract

This research investigates the connectivity properties of the k-distance product graph in order to determine its connectivity κ(G) and edge connectivity κ′(G). The study is motivated by fundamental concepts in graph theory, including connected graphs, connectivity, and edge connectivity. Let G = (V, E) be a graph. If G is not a complete graph, the connectivity of G is defined as the minimum number of vertices whose removal disconnects G. If G is a complete graph, its connectivity is defined as κ(G) = |V| − 1. The vertex connectivity of G is denoted by κ(G). Similarly, for a graph G = (V, E) with |V| > 1, the edge connectivity of G is defined as the minimum number of edges whose removal disconnects the graph. If |V| = 1, the edge connectivity is defined to be 0. The edge connectivity of G is denoted by κ′(G).

Let G and H be graphs. The k-distance product of G and H, denoted by $G\square_k H$, is defined as follows.

The vertex set is $V(G\square_k H) = V(G) \times V(H)$.

The edge set is $E(G\square_k H) = \{(u, x), (v, y) | x = y \text{ and } 1 \leqslant d_G(u, v) \leqslant k, \text{ or } u = v \text{ and } 1 \leqslant d_H(x, y) \leqslant k\}$.

That is, two vertices (u, x) and (v, y) are adjacent in $G\square_k H$ if either the second components are equal and the distance between u and v in G is at most k, or the first components are equal and the distance between x and y in H is at most k.

Preliminaries

Definition 1 : Let G be a graph. We say that G is a **connected graph** if, for every pair of vertices u and v in G, there exists at least one path connecting them. Alternatively, this means it is possible to travel from any vertex to any other vertex in the graph by traversing its edges. If there exists at least one pair of vertices between which no path exists, the graph is called a **disconnected graph**.

Definition 2 : Let G = (V, E) be a graph, and let r = {u, v} ∈ E. We say that the edge r is **incident** to the vertices u and v.

Definition 3 : Let G = (V, E) be a graph and U ⊆ V. We define G - U as the graph obtained by deleting all vertices in U from V and deleting all edges incident to vertices in U from E. Let F ⊆ E. Define G - F as the graph obtained by deleting all edges in F from E.

Definition 4 : Let G = (V, E) be a graph and let u ∈ V(G). The set of vertices adjacent to u in G is called **the neighborhood** of u, denoted by

$$N_G(u) = \{v \in V(G) \mid uv \in E(G)\}.$$

The number of edges incident with u is called the degree of u. (In the case that an edge is a loop, it is counted twice) The degree of u is denoted by $d_G(u)$.

Definition 5 : **A complete graph** with n vertices, denoted by K_n , is a simple graph in which every pair of distinct vertices is connected by exactly one edge. That is, let $K_n = (V, E)$ be a simple graph where |V| = n, and for every pair of distinct vertices u, v ∈ V, there exists an edge uv ∈ E.

Definition 6 : Let G be a graph. If G is not a complete graph, we define **the connectivity** of G as the minimum number of vertices whose removal disconnects G. If G is a complete graph, its connectivity is defined as |V| - 1. We denote the connectivity of G by the symbol **K**(G).

Definition 7 : Let G = (V,E) be a graph. If $|V| \neq 1$, we define **the edge connectivity** of G as the minimum number of edges that must be removed from G so that the remaining graph is disconnected. If |V| = 1, then the edge connectivity of G is defined to be 0. We denote the edge connectivity of G by the symbol K'(G).

Definition 8 : Let G = (V,E) be a graph and let u,v ∈ V(G). The distance between the vertices u and v in the graph G, denoted by $d_G(u,v)$, is defined as the length of the shortest path connecting u and v. If there is no path between u and v, then **the distance** $d_G(u,v)$ is defined to be infinite.

Definition 9 : Let G and H be graphs.
The Cartesian Product of graph of G and H , denoted by $G \square H$, is the graph whose vertex set is $V(G \square H) = V(G) \times V(H)$ and whose edge set is defined by $E(G \square H) = \{(u,x), (v,y) \mid (u=v \text{ and } \{x,y\} \in E(H)) \text{ or } (x=y \text{ and } \{u,v\} \in E(G))\}$.

Definition 10 : Let G and H be graphs.
The k-distance product of graphs G and H, denoted by $G \square_k H$,
is the graph whose vertex set is

$$V(G \square_k H) = V(G) \times V(H)$$

and whose edge set is defined by

$$E(G \square_k H) = \{(u,x)(v,y) \mid (x = y \text{ and } 1 \leqslant d_G(u,v) \leqslant k) \text{ or } (u = v \text{ and } 1 \leqslant d_H(x,y) \leqslant k)\}.$$

Objective

- To determine bounds of the connectivity $\kappa(G\square_k H)$.
- To determine bounds of the edge connectivity $\kappa'(G\square_k H)$.
- To extend classical connectivity results of the Cartesian product graph to the k-distance product graph.

Results

Theorem 1: (Degree of a Vertex in the k-Distance Product Graph)
Let G and H be connected graphs, and let k be a positive integer. For every vertex $(u,x) \in V(G\square_k H)$ the degree of (u,x) in $G\square_k H$ is given by

$$d_{G\square_k H}(u,x) = |\{v \in V(G) \mid 1 \leqslant d_G(u,v) \leqslant k\}| + |\{y \in V(H) \mid 1 \leqslant d_H(x,y) \leqslant k\}|.$$

That is the degree of the vertex (u,x) in $G\square_k H$ equals the number of vertices in G whose distance from u is between 1 and k, plus the number of vertices in H whose distance from x is between 1 and k.

Theorem 2 : (Upper Bound on the Connectivity of the k-Distance Product Graph)
Let G and H be connected graphs, and let k be a positive integer. Then

$$\kappa(G\square_k H) \leqslant \delta(G\square_k H).$$

Theorem 3 : (Relationship between Connectivity, Edge Connectivity, and Minimum Degree)
Let G and H be connected graphs, and let k be a positive integer. Then

$$\kappa(G\square_k H) \leqslant \kappa'(G\square_k H) \leqslant \delta(G\square_k H).$$

Theorem 4 : (Monotonicity of Connectivity with Respect to k)
Let G and H be connected graphs. Let k_1 and k_2 be positive integers such that $k_1 \leqslant k_2$. Then

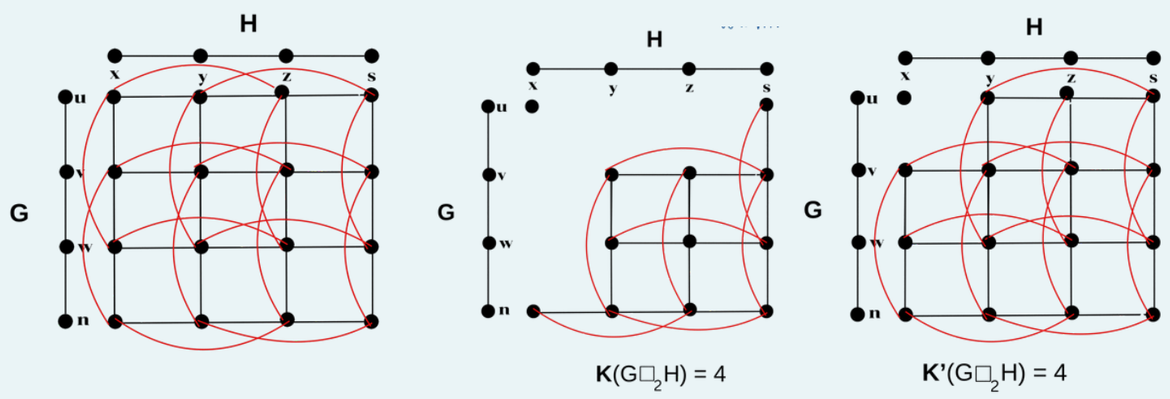
$$\kappa(G\square_{k1} H) \leqslant \kappa(G\square_{k2} H)$$
$$\text{and } \kappa'(G\square_{k1} H) \leqslant \kappa'(G\square_{k2} H).$$

Conclusion

In this independent study, the properties of vertex connectivity and edge connectivity of the k-distance product graph were investigated. This concept extends the idea of the Cartesian product of graphs by incorporating distance conditions in the component graphs, providing greater flexibility in the construction of product graphs.

The results show that the degree of a vertex in the graph $G\square_k H$ can be directly determined from the number of vertices within distance at most k in the component graphs. Furthermore, it was found that the k-distance product of two connected graphs remains connected, and the connectivity of the resulting graph tends to increase or remain unchanged as the value of k increases.

Examples of k-Distance Product Graphs



Reference

[1] วรรณศิริ วรรณสิทธิ์, เอกสารคำสอนวิชา 206281 วิยุตคณิต, ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่
[2] West, D. B. (2001). Introduction to graph theory (2nd ed.). Prentice Hall.