Sequences in Pythagorean Triples

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Abstract

This independent study explores the relationship between numerical sequences and the Pythagorean formula by applying Martin's (1875) Pell-type formula to the Fibonacci sequence. The results show that the formula is not limited to Pell sequence, but reflectsthe Fibonacci sequence. Moreover, we give an example showing that the formula can not be applied to some sequence. This study establishes a structured method for generating Pythagorean numbers and clarifies the connection between number theory and geometry.

Objectives

1. To study the relationship between numerical sequences, particularly the Fibonacci sequence and the Pell sequence, and the Pythagorean formula.

2. To examine the conditions under which terms of the Fibonacci and Pell sequences can be used to generate Pythagorean numbers.

Methodology

1. Fundamental concepts related to numerical sequences are reviewed, with particular emphasis on the Fibonacci sequence and the Pell sequence. This includes their definitions, recurrence relations, and basic properties relevant to number theory and the construction of Pythagorean numbers.

2. The formula presented by Martin (1875) is applied to selected terms of the Fibonacci sequence and the Pell sequence. The resulting numerical values are computed and examined in order to construct corresponding Pythagorean triples.

3. The conditions under which terms of the Fibonacci and Pell sequences satisfy the Pythagorean formula are analyzed. This step aims to identify similarities and differences in the way both sequences generate Pythagorean numbers using the same construction method.

4. The obtained results are verified through algebraic analysis and illustrative examples.

The findings are then systematically organized and presented in a written report to clearly demonstrate the relationships between the Fibonacci sequence, the Pell sequence, and Pythagorean triples.

Preliminaries

- A Pythagorean triple is a triple (a, b, c) of positive integers such that

  For example (3, 4, 5), (7 ,12 ,13) are Pythagorean triple.

- The Fibonacci sequence is the sequence of integers defined by

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2$$

  The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, . . .

- The Pell numbers is the sequence of integers defined by

$$P_1 = 1, P_2 = 2, P_n = 2P_{n-1} + P_{n-2}, \quad \text{for } n \geq 3$$

  The first few Pell numbers are 1, 2, 5, 12, 29, 70, . . .

- (Binet's Formula) The Fibonacci numbers admit the following explicit formula (originally derived by Jacques Philippe Marie Binet (1843)):

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0, \quad \text{where,} \quad \alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

- The n th Pell number Pn is given explicitly by the Binet's formula The explicit formula is analogous to Binet's formula, named after Jacques Philippe Marie Binet.

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}, n \geq 0.$$

- The Pell-Lucas sequence is the sequence of integers defined by

$$Q_1 = 2, \quad Q_2 = 6, \quad Q_n = 2Q_{n-1} + Q_{n-2} \text{ for } n > 2$$

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Main Results

- **Lemma 1:** Let x and y be positive integers. Then $(2xy,\ y^2 - x^2,\ x^2 + y^2)$ is Pythagorean triple.

- **Theorem 2 :**

  If P_n and P_{n+1} are consecutive Pell numbers, then $P_n^2 + P_{n+1}^2$ is Pell number.

Table 3.2: Pythagorean triples generated from consecutive Pell numbers.

n	P_n	P_{n+1}	$a = 2P_nP_{n+1}$	$b = P_{n+1}^2 - P_n^2$	$C = \sqrt{a^2+b^2}$
0	0	1	0	1	1
1	1	2	4	3	5
2	2	5	20	21	29
3	5	12	120	119	169
4	12	29	696	697	985
5	29	70	4060	4059	5741
6	70	169	23660	23661	33461
7	169	408	137904	137903	195025
8	408	985	803760	803761	1136689
9	985	2378	4684660	4684659	6625109
10	2378	5741	27304196	27304197	38613965

- **Theorem 3:**

  If F_n and F_{n+1} are consecutive Fibonacci numbers, then $\mathrm{F}_n^2 + F_{n+1}^2$ is Fibonacci number.

Table 3.3: Fibonacci-generated Pythagorean triples and index pattern

n	F_n	F_{n+1}	$C = F_n^2 + F_{n+1}^2$	Check	n	$F_n(c)$
1	1	1	2	Yes	1	3
2	1	2	5	Yes	2	5
3	2	3	13	Yes	3	7
4	3	5	34	Yes	4	9
5	5	8	89	Yes	5	11
6	8	13	233	Yes	6	13

If $x = F_n$ and $y = F_{n+1}$ that is, $c = x^2 + y^2 = F_n^2 + F_{n+1}^2$. This pattern suggests the identity : $F_n^2 + F_{n+1}^2 = F_{2n+1}$,

- **Remark :** Consider Pell–Lucas terms as follows:

$$a = 2Q_nQ_{n+1}, \quad b = Q_{n+1}^2 - Q_n^2, \quad c = \sqrt{a^2+b^2}.$$

Table 3.5: Pythagorean construction from two consecutive Pell-Lucas

n	Q_n	Q_{n+1}	$a = 2Q_nQ_{n+1}$	$b = Q_{n+1}^2 - Q_n^2$	$c = \sqrt{a^2+b^2}$	Check
0	2	2	8	0	8	No
1	2	6	24	32	40	No
2	6	14	168	160	232	No
3	14	34	952	960	1352	No
4	34	82	5576	5568	7880	No
5	82	198	32472	32480	45928	No

From Table 3.5, c is not an element of the Pell–Lucas sequence. Therefore, the method fromLemma 1 fails to generate Pythagorean triples, and c does not belong o the sequence .

Recommendations

1. Not every sequence works with this formula. Before applying the Pell-type formula, the structure of the sequence should be examined to determine whether it is compatible with the Pell or Fibonacci framework.

2. Test the formula on other sequences. Future research may apply the formula to other second-order linear recurrence sequences to investigate whether they can also generate Pythagorean triples.

3. Study the cases where the formula does not work. Analyzing sequences that do not satisfy the formula may help clarify its limitations and structural constraints.

4. Develop a more general idea beyond one formula. Instead of relying solely on the Pell-type formula, future work could aim to construct a broader theoretical framework connecting numerical sequences and Pythagorean structures.