A simple solution of the Bratu problem

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Abstract

This independent study investigates the solution of the one-dimensional Bratu's problem, which is a nonlinear ordinary differential equation, by applying the finite difference method. The discretization process leads to a system of nonlinear equations, which is solved using Newton's iterative method. Numerical approximations are obtained and compared with the analytical solution to evaluate the accuracy of the proposed approach. The results show that the finite difference method can approximate the solution of the one-dimensional Bratu's problem efficiently and with high accuracy.

Introduction

$$\frac{d^2u}{dx^2} + Ce^u = 0, \quad 0 \le x \le 1$$
$$u(0) = u(1) = 0$$

The **Bratu's equation** is a second-order nonlinear ordinary differential equation that is widely used as a benchmark problem for testing numerical methods. The behavior of its solutions depends on the value of the parameter C. This independent study investigates the one-dimensional Bratu problem under Dirichlet boundary conditions.

Methodology

Finite difference method

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + Ce^{u_j} = 0 \qquad ; j = 1, 2, \dots$$

the second derivative is approximated by the central difference formula to transform a differential equation into a system of nonlinear equations.

Newton's iterative method

newton's iteration formula at step k

$$J\left(\mathbf{u}^{(k)}\right)\Delta\mathbf{u}^{(k)} = -F\left(\mathbf{u}^{(k)}\right),$$
$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \Delta\mathbf{u}^{(k)}$$

(Where J is the Jacobian matrix and F is the nonlinear function vector)

Summary

- Initial Guess: The initial approximation is set as $u^{(0)} = 0$ for all internal points.
- Stopping Criterion: The iterative process is repeated until the change $\Delta u^{(k)}$ is smaller than a prescribed tolerance ϵ .

Conclusion

This study examines the one-dimensional Bratu's equation under Dirichlet boundary conditions using both analytical and numerical approaches. The numerical solution, obtained via a finite difference discretization with Newton's method, agrees well with the analytical solution, and the error decreases as the grid is refined. As the parameter approaches the critical regime, the solution becomes more nonlinear and the numerical error tends to increase.

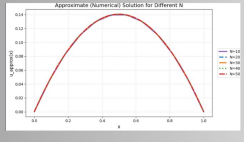
Numerical Results

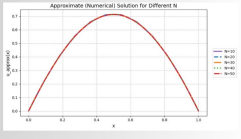
the exact solution $u(x) = 2\ln\left(\frac{\cosh\alpha}{\cosh(\alpha(1-2x))}\right)$

where α satisfies the transcendental equation ; $\cosh\alpha = \frac{4}{\sqrt{2C}}\alpha$

Compare the numerical approximations for N = 10, 20, 30, 40, 50

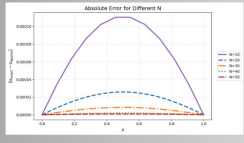
Case i : C = 1 **Case ii : C = π**

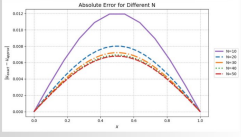




A plot of the numerical solutions computed using the finite difference method

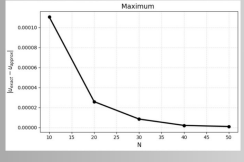
All numerical solutions exhibit a curved profile that is perfectly symmetric around the midpoint x = 0.5 . The solutions satisfy the Dirichlet boundary conditions, reaching zero at both x = 0 and x = 1 .

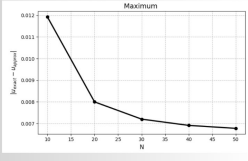




A plot of the absolute error

The graph of the absolute error for C = π. It can be clearly seen that the overall error is higher than in the case C = 1 . This is because when C is close to the critical value, the resulting solution becomes more highly curved, causing the approximation using the finite-difference method to have larger errors accordingly. Nevertheless, the trend of decreasing error as N increases is still clearly observed.





A plot of the maximum absolute error

When C = π the maximum absolute error is on the order of 10^{-3}, which is significantly larger than the case C = 1 where it is around 10^{-4} to 10^{-6} . This happens because increasing C makes the term Ce more nonlinear and more strongly curved, which increases the truncation error of the finite-difference formulas. Moreover, as N increases, the error decreases more slowly for C = π than for C = 1 , so when C is near a critical value, a larger N is required to achieve the same accuracy.

Reference

A. Mohsen, A simple solution of the Bratu problem, Computers & Mathematics with Applications, 2014; 67(1): 26–33.