Sahoo- and Wayment-type integral mean value theorems

Napatson Intharos Student ID: 650510516

Advisor: Assoc. Prof. Dr.Somlak Utudee

Department of Mathematics, Faculty of Science, Chiang Mai University

ABSTRACT

The integral mean value theorem states that if f is a real-valued function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) , then there exists a point $c \in (a,b)$ such that $f(c) = \frac{\int_a^b f(s)ds}{b-a}$. The proof applies Rolle's Theorem and the Fundamental theorem of calculus. The objective of this independent study is to extend the Sahoo- and Wayment-type integral mean value theorems to functions in the class $\mathcal{F}$, which are continuous on the open interval (a,b) and possess limits at the endpoints but are not necessarily continuous on the closed interval. This study investigates several integral mean value theorems for functions in the class $\mathcal{F}$, together with illustrative examples. In the proofs, auxiliary functions are constructed and important mean value theorems are applied, along with the Fundamental theorem of calculus and Leibniz's rule, to determine a point $c \in (a,b)$ at which the function value is related to the integral mean value.

FLETT'S THEOREM & WAYMENT'S THEOREM

Flett's theorem : Let $f : [a,b] \to \mathbb{R}$ be differentiable on the close interval $[a,b]$ if $f'(a) = f'(b)$. Then there exists a point $c \in (a,b)$ such that

$$f(c) - f(a) = f'(c)(c-a)$$

Wayment's theorem : Let $f : [a,b] \to \mathbb{R}$ and $f(a) = f(b)$,Then there exists at least one point such that $c \in (a,b)$

$$f(c)(c-a) = \int_a^c f(s)ds$$

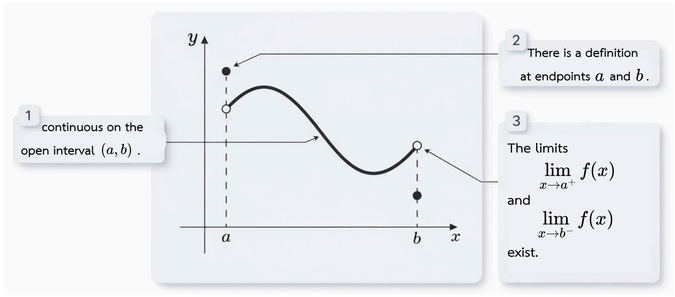
CLASS $\mathcal{F}$

Definition : Let $f : [a,b] \to \mathbb{R}$ We say that f belongs to the class $\mathcal{F}$ (denoted $f \in \mathcal{F}$) if the following conditions hold:

1. f is a real-valued function continuous on the open interval (a,b) .
2. f is defined at both endpoints a and b .
3. The limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

From the above conditions, we observe :

1. The limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ need not equal $f(a)$ and $f(b)$, respectively.
2. If $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$,then $f \in \mathcal{F}$ is continuous on the closed interval $[a,b]$.



From the definition of class F functions, it can be seen that these functions are discontinuous at their endpoints, but they still possess integration properties that relate to continuous functions. Next, we will explore and analyze important mathematical theorems that primarily concern continuity conditions on closed intervals, and then apply them to functions in class F.

Preliminary Integrable function : Let $f \in \mathcal{F}$ and $g : [a,b] \to \mathbb{R}$ where

$$g(x) = \begin{cases} f(x), & x \in (a,b) \\ \lim_{x\to a^+} f(x), & x = a \\ \lim_{x\to b^-} f(x), & x = b \end{cases}$$

We get, g is continuous on the closed interval $[a,b]$.

Thus $\int_a^b f(x)\,dx$ exist and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

Preliminary Mean value theorem : Let $f : [a,b] \to \mathbb{R}$ and differentiable on the open interval (a,b) , then there exists a point $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a)$$

Preliminary Intermediate value theorem : Let $f : [a,b] \to \mathbb{R}$ and $f(a) \neq f(b)$ If d is any real number between $f(a)$ and $f(b)$, then there exists at least one real number in the open interval (a,b) such that

$$f(c) = d$$

Preliminary Mean value theorem for integral : Let $f : [a,b] \to \mathbb{R}$ and differentiable on the open interval (a,b) , then there exists a point $c \in (a,b)$ such that

$$f(c) = \frac{\int_a^b f(s)ds}{b-a}$$

SYMBOLS

Next, we will introduce the symbols used in this context. Let $f : [a,b] \to \mathbb{R}$,where $f \in \mathcal{F}$

$$M(f) = \frac{1}{2}\int_a^b f(s)\,ds,$$

$$I_1(f) = \frac{1}{b-a}\int_a^b (b-s)f(s)\,ds,$$

$$I_2(f) = \frac{1}{b-a}\int_a^b (s-a)f(s)\,ds.$$

All the equations of Theorem 1-14 are constructed from these three forms of integral mean relationships.

RESULT

Let $f \in \mathcal{F}$, there exists a point $c \in (a,b)$ such that the integral mean equation in the following forms is true under the conditions specified in each theorem.

condition : $M(f) = I_1(f)$

Theorem 1 (Basic Equation)

$$(c-a)f(c) = \int_a^c f(s)ds$$

Theorem 2 (General Form)

$$f(c) = \frac{\int_a^c f(s)ds}{c-a} + \frac{6[M(f) - I_1(f)]}{(b-a)^2}(c-a)$$

Theorem 3 (Basic Equation)

$$f(c) = \frac{\int_c^b f(s)ds}{b-c}$$

Theorem 4 (General Form)

$$f(c) = \frac{\int_c^b f(s)ds}{b-c} + \frac{6[M(f) - I_1(f)]}{(b-a)^2}(b-c)$$

condition : $2M(f) = I_1(f)$

Theorem 5 (Basic Equation)

$$f(c) = \frac{\int_c^a f(s)ds}{b-a}$$

Theorem 6 (General Form)

$$f(c) = \frac{\int_a^b f(s)ds}{b-a} + \frac{6\,[2M(f) - I_1(f)]}{(b-a)^3}[(2c-a-b)(b-a) + (c-a)(b-c)]$$

Theorem 7 (Basic Equation)

$$f(c) = \frac{\int_a^b f(s)ds}{b-c}$$

Theorem 8 (General Form)

$$f(c) = \frac{\int_c^b f(s)ds}{b-c} + \frac{6\,[2M(f) - I_1(f)]}{(b-a)^2}(2c-a-b)$$

condition : $2M(f) = I_2(f)$

Theorem 9 (Basic Equation)

$$f(c) = \frac{\int_a^b f(s)ds}{c-a}$$

Theorem 10 (General Form)

$$f(c) = \frac{\int_a^b f(s)ds}{c-a} - \frac{6\,[2M(f) - I_2(f)]}{(b-a)^2}(2c-a-b)$$

Theorem 11 (Basic Equation)

$$f(c) = \frac{\int_c^b f(s)ds}{b-a}$$

Theorem 12 (General Form)

$$f(c) = \frac{\int_c^b f(s)ds}{b-a} + \frac{6\,[2M(f) - I_2(f)]}{(b-a)^3}[(b-c)(c-a) - (2c-a-b)(b-a)]$$

Independent relationship

Theorem 13 : Let $f \in \mathcal{F}$ .Then there is a point $c \in (a,b)$ such that $f(c) = \frac{\int_c^b f(s)ds}{c-a}$.

Theorem 14 : Let $f \in \mathcal{F}$ .Then there is a point $c \in (a,b)$ such that $f(c) = \frac{\int_a^c f(s)ds}{b-c}$.

CONCLUSION

The Sahoo and Wayment theorems provide a better understanding of the relationship between function values and integral means, and can be applied to a broader class $\mathcal{F}$ than the original mean theorem.

REFERENCE

[1] Aydin Tiryaki, Sahoo- and Wayment-type integral mean value theorems, TheAmerican Mathematical Monthly (2010)

[2] วา รุ นันท์ อิน ถา ก้อน, เอกสาร ประกอบ การ สอน กระบวน วิชา 206111 แคลคูลัส 1, ภาค วิชาคณิตศาสตร์ มหาวิทยาลัยเชียงใหม่, ภาคการศึกษา 1/2565

[3] Zou Yunzhi, Single Variable Calculus, A First Step, Walter de Gruyter GmbH and CoKG, 2018

[4] Weir Maurice D., Joel Hass, and George Brinton Thomas. Thomas' Calculus Multivari-able Addison-Wesley, 2010