The independent number of the generalized lexicographic product of a path graph and cycle graphs

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Abstract

Gave a graph $G=(V,E)$, a set $S \subseteq V$ is an **independent set** if for every $u,v \in S$, the edge $\{u,v\} \notin E$. The **independence number** $(\alpha(G))$ of G is defined as the maximum cardinality of an independent set. And let $(H_x)_{x\in G}$ be graphs with $H_x=(V_x, E_x)$. The **generalized lexicographic product** of G and $(H_x)_{x\in G}$ is defined to be $G[(H_x)_{x\in G}]:=(\{(x,y_x)|x\in V, y_x\in H_x\}, \{((x,y_x),(x',y'_x))|(x,x')\in E\} \cup \{((x,y_x),(x,y'_x))|x\in V,(y_x,y'_x)\in E_x\})$. In this independent study, we are interested in determining the independence number of the generalized lexicographic product of a path graph and cycle graphs.

Preliminaries

Definitions 1: A **graph** G consists of two sets, namely V(G) and E(G), where

- V(G) is a finite and non-empty set.
- E(G) is a set of ordered pairs of elements in V(G).

The elements of V(G) are called vertices, and the elements of E(G) are called edges.

The graph G can be denoted by G = (V(G), E(G)).

Remark: The set E(G) may contain repeated elements (parallel edges or multi-edges).

If $\{u,v\} \in E(G)$, then u and v are called the **endpoints** of the edge $\{u,v\}$.

For convenience, the edge $\{u,v\}$ is often written as uv.

Definitions 2: Let u and v be vertices in a graph. We say that the two vertices are **adjacent** if they are the endpoints of the same edge. Similarly, we say that two edges are adjacent if they share a common endpoint.

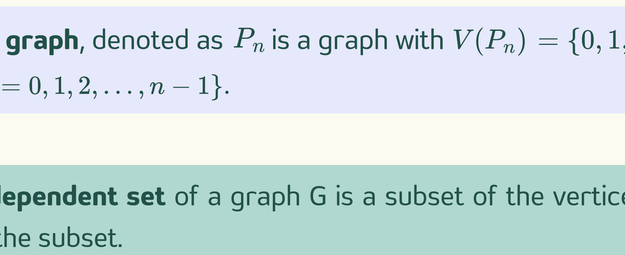
Definitions 3: A **cycle graph**, denoted as C_n is a graph with $V(C_n)=\{0,1,2,\ldots,n-1\}$ and $E(C_n)=\{\{i,i+1\}|i=0,1,2,\ldots,n-2\}\cup\{\{n-1,0\}\}$.

Definitions 4: A **path graph**, denoted as P_n is a graph with $V(P_n)=\{0,1,2,\ldots,n-1\}$ and $E(P_n)=\{\{i,i+1\}|i=0,1,2,\ldots,n-1\}$.

Definitions 5: An **independent set** of a graph G is a subset of the vertices such that no two adjacent vertices in the subset.

Definitions 6: The **independence number** $(\alpha(G))$ of G is defined as the maximum cardinality of an independent set.

Definitions 7: The **lexicographic product** of G_1 and G_2 is defined to be $G_1[G_2]:=(V_1\times V_2, \{((x,y),(x',y'))|(x,x')\in E_1\}\cup\{\{(x,y),(x,y')\}|x\in V_1,(y,y')\in E_2\})$.



$P_3[C_4]$

Definitions 8: Let $G=(V,E)$ and let $(H_x)_{x\in G}$ be graphs with $H_x=(V_x,E_x)$. The **generalized lexicographic product** of G and $(H_x)_{x\in G}$ is defined to be $G[(H_x)_{x\in G}]:=(\{(x,y_x)|x\in V,y_x\in H_x\},\{((x,y_x),(x',y'_x))|(x,x')\in E\}\cup\{((x,y_x),(x,y'_x))|x\in V,(y_x,y'_x)\in E_x\})$.

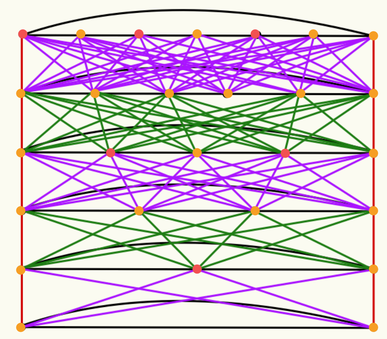
Objectives

- To study the properties of the generalized lexicographic product of graphs.
- To determine the independence number of the generalized lexicographic product of a path graph and cycle graphs.

Result

- The Independence Number of the Generalized Lexicographic Product of P_m and $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ when m is even.

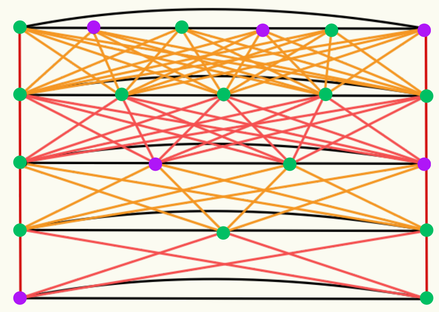
Theorem 1: Let m be an even positive integer and let $n_1, n_2, \ldots, n_m$ be positive integers such that $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_m$. And let $G=P_m[C_{n_1}, C_{n_1}, \ldots, C_{n_m}]$ be the generalized lexicographic product, where m is even and $n_k \geqslant 2$ for all k. Then the independence number of the graph G is $\alpha(G)=\sum_{k=1}^{\lceil \frac{m}{2} \rceil} \lfloor \frac{n_k}{2} \rfloor$.



$P_6[(C_2, C_3, C_4, C_5, C_6, C_7)]$

- The Independence Number of the Generalized Lexicographic Product of P_m and $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ when m is odd.

Theorem 2: Let m be an odd positive integer and let $n_1, n_2, \ldots, n_m$ be positive integers such that $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_m$. And let $G=P_m[C_{n_1}, C_{n_1}, \ldots, C_{n_m}]$ be the generalized lexicographic product, where m is odd and $n_k \geqslant 2$ for all k. Then the independence number of the graph G is $\alpha(G)=\sum_{k=0}^{\lceil \frac{m}{2} \rceil} \lfloor \frac{n_k}{2} \rfloor$.



$P_5[(C_2, C_3, C_4, C_5, C_6)]$

Conclusion

This independent study investigates the independence number of the generalized lexicographic product of a path graph and cycle graphs, where each vertex of the path graph is replaced by a cycle graph. The results show that the independence number of the resulting graph depends on the structure of the path graph and the number of vertices in the cycle graph. In particular, the cases where the cycle graph has an even or odd number of vertices produce different structures of maximum independent sets. These findings provide insight into the behavior of independence numbers under generalized lexicographic product constructions and may serve as a foundation for studying other properties of graph products.

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