Deviation of Voronoi cell in the Regular Tessellations on the Plane

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1 Abstract

This independent study investigates the construction of a regular hexagonal tessellation by interpreting the regular hexagonal tiling from the perspective of a Voronoi diagram, in which each regular hexagon is regarded as a cell. The study focuses on the deviation patterns of Voronoi cells within a regular tessellation. Beginning with the specification of the generator points of the Voronoi diagram, the research examines how the polygon in the Voronoi diagram change when the selected generator point is displaced in a prescribed direction on the plane. The findings reveal relationships between the position of the generator point and the number of sides of the corresponding Voronoi cell.

2 Problem statement

This study focuses on Voronoi cell patterns that deviate from regular tessellation, as well as the geometric properties of tessellation and Voronoi diagrams on a plane.

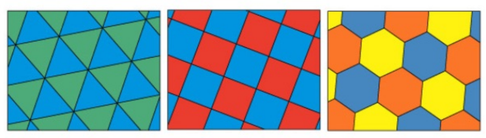
3 Basic Mathematical Concepts

A **tessellation** is a collection of shapes that fit together without gaps or overlap to cover the infinite mathematical plane.

For the plane S_n and shapes P_i it holds that $S_n \subseteq \bigcup_{i=1}^{n} P_i$ and $Int\,(P_i) \cap Int\,(P_j) \neq \varnothing$, where $Int\,(P_i)$ denotes the interior of shapes P_i

A polygon is said to be regular polygon if it consists of equal length of edges, and all adjacent angle of edges are equal

A **regular tessellation** is an edge-to-edge tessellation by congruent regular polygons. There are exactly three regular tessellations, based on the equilateral triangle, square, and regular hexagon.



Trlangles, Squares and Hexagons

Voronoi Diagram

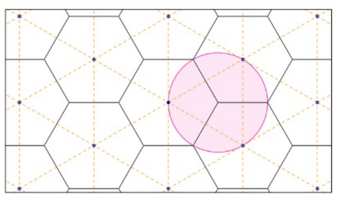
Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of n sites on the plane. The **Voronoi region** of the site $p_i \in S$ is defined by

$$Vor\,(p_i) = \{x \in \mathbb{R}^2 \,|d\,(x, p_i) \leqslant d\,(x, p_j)\}$$

for all points p_1, p_2 in set S, where $i \neq j$, and $d\,(x, y)$ denotes the Euclidean distance between points x and y in the plane.

Regular hexagon Tessellation in terms of Voronoi diagram

we can consider a regular hexagonal tessellation in terms of voronoi diagram by focusing on the centroid of each hexagon. This point is regarded as a generator of voronoi cell.

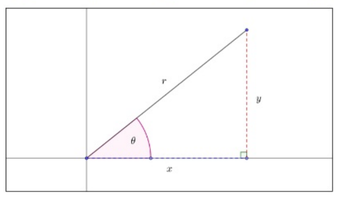


Empty Circle Property of a Voronoi vertex

A vertex v of a Voronoi diagram is the center of a circle that passes through at least three generator points of S, with no other points from S lying inside that circle.

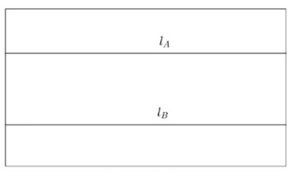
Polar coordinates

For each point (x, y) in rectangular coordinate, the polar coordinate of point (x, y) is represented by $x = r\cos\theta$,$x = r\cos\theta$ $r = \sqrt{x^2 + y^2}$ and $\theta = arctan\left(\frac{y}{x}\right)$

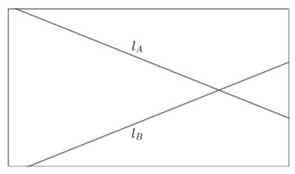


Polygon induced by voronoi diagram.

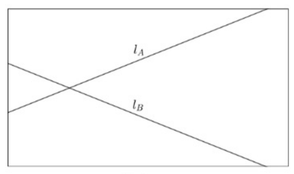
Assume that edges of a polygon is formed by perpendicular bisectors of voronoi generators. If there are straight lines l_A and l_B acting as two perpendicular bisectors, the intersection of these perpendicular bisectors can result in the vertices of a polygon according to the following cases:

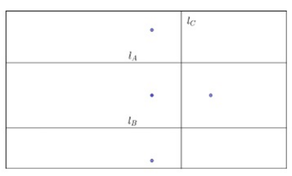


$l_A \,/\!/\, l_B$



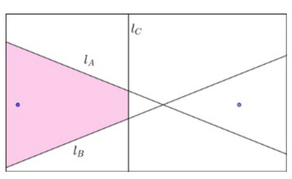
l_A intersects with l_B on one side.



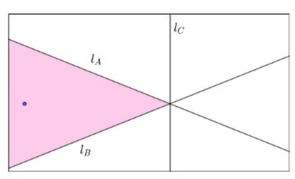


In the case of 3 lines

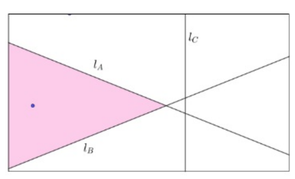
l_A is parallel to l_B, and there is an l_C that is not parallel to l_A and l_B.



(A)



(B)



(C)

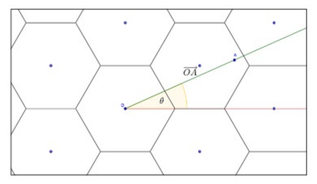
1) l_A is not parallel to l_B, and there is a line l_C that is not parallel to l_A and l_B. In (A), l_C intersects both l_A and l_B, forming a vertex of the polygon and resulting in an increase in the total number of vertices.

2) In (B) and (C), l_C intersects l_A and l_B starting from the intersection point of l_A and l_B, extending through the area enclosed by l_A and l_B.

4 Method

We consider the transformation of Voronoi cells when the generator points are shifted from their original positions, which correspond to the locations in a regular hexagonal tessellation. By leveraging the symmetry of regular hexagons.

Let T represent a Voronoi cell of a regular hexagonal tessellation surrounded by other regular hexagons.



Step 1: Place the point O at the same position as the generator point of cell T and fix its position.

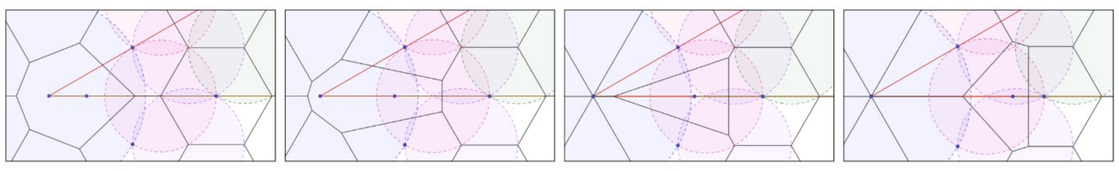
Step 2: Construct a ray $\overrightarrow{OA}$ along the positive x-axis, where θ represents the angle formed between ray $\overrightarrow{OA}$ and the x-axis.

In analyzing the deviation of the cell, attention is focused on the behavior of the shape transformation of the polygon, specifically how it changes from an n-gon to a k-gon.

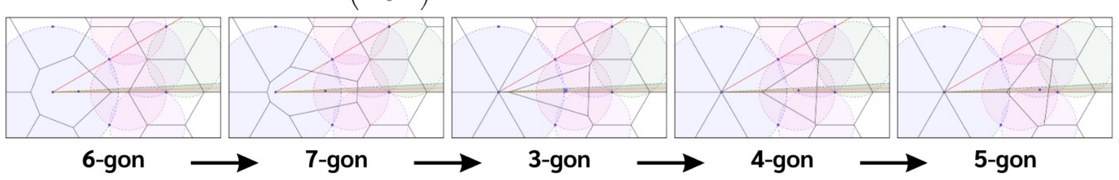
5 Results

The deviation of cells when moving the generator point along the ray $\overrightarrow{OA}$ in a constant direction within the angular range $0^\circ \leqslant \theta \leqslant 30^\circ$. A change in shape is observed from a regular hexagon to an irregular hexagon and other polygons, depending on the orientation of the perpendicular bisectors of the generator point relative to other generator points. The different cases of deviation can be classified into 6 cases.

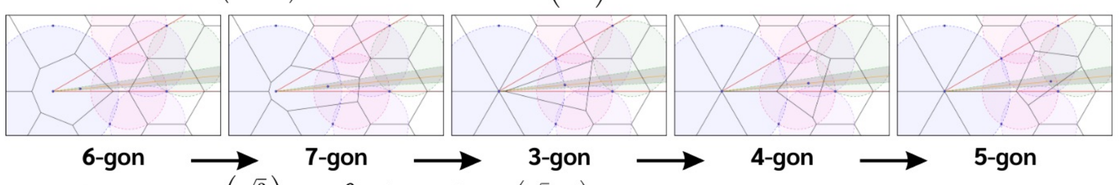
Case 1: $\theta = 0^\circ$



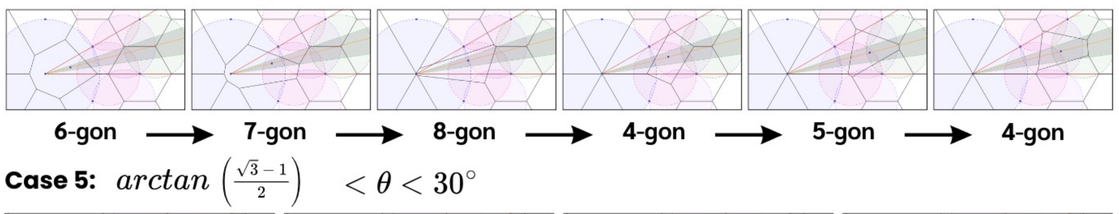
Case 2: $0^\circ < \theta < arctan\left(\frac{2-\sqrt{3}}{5}\right)$



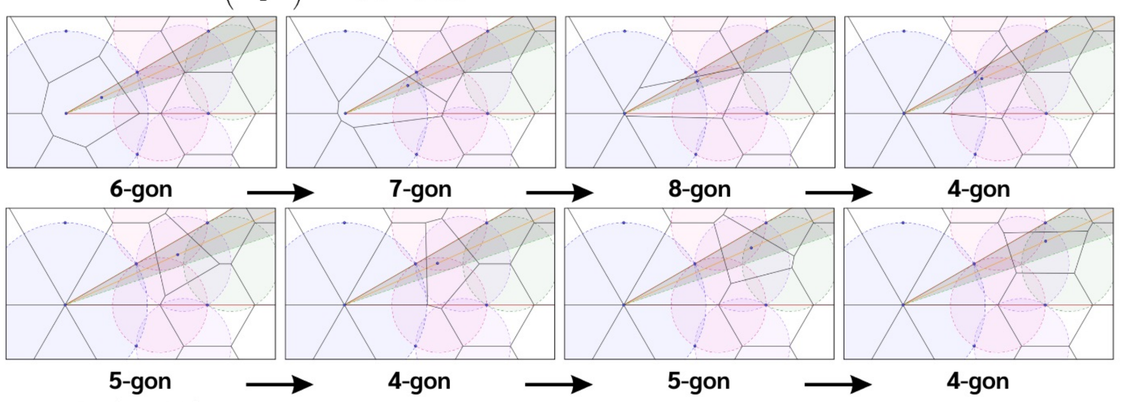
Case 3: $arctan\left(\frac{2-\sqrt{3}}{5}\right) \leqslant \theta \leqslant arctan\left(\frac{\sqrt{3}}{9}\right)$



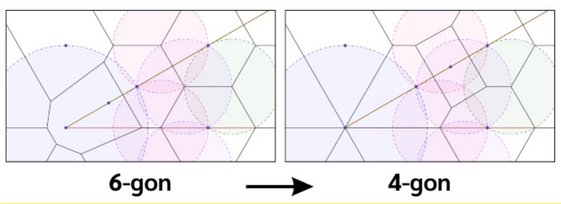
Case 4: $arctan\left(\frac{\sqrt{3}}{9}\right) < \theta \leqslant arctan\left(\frac{\sqrt{3}-1}{2}\right)$



Case 5: $arctan\left(\frac{\sqrt{3}-1}{2}\right) < \theta < 30^\circ$



Case 6: $\theta = 30^\circ$



6 Conclusion

In this study, polygons are considered as regions defined by the intersection of multiple half-planes. The geometric transformation of a polygon depends on the relative position of the generator of interest with respect to its neighboring generators. We examine how the number of sides of the polygon changes under such positional variations and present several observations describing these transitions. The proofs require a careful analysis of the relative configurations among the generating points. This approach provides useful insight into the structural regularity of a given tessellation and may contribute to the classification of tessellations in future research.

References

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