Adomian decomposition method for solving Bratu equation

Preeyaporn Khamwiangsa Student ID : 650510521
Assoc. Prof. Dr. Morrakot Khebchareon
Department of Mathematics, Faculty of Science, Chiang Mai University

Abstract

This independent study aims to investigate analytical solutions of nonlinear ordinary differential equations of the Bratu type using the Adomian Decomposition Method (ADM).This method has the advantage of decomposing nonlinear operators into Adomian polynomials, which facilitates the analytical treatment of nonlinear terms.The ADM is capable of providing closed-form analytical solutions efficiently, with high accuracy and reliability.

Introduction

Differential equations are widely used to model phenomena in science and engineering, particularly nonlinear equations arising in many physical processes. A well-known example is the Bratu equation, a second-order nonlinear differential equation. In this study, the Adomian Decomposition Method (ADM) is applied to obtain approximate analytical solutions.

Objective

1. To investigate numerical solution methods for nonlinear ordinary differential equations.
2. To analyze the Adomian Decomposition Method for solving nonlinear equations.
3. To determine solutions of the Bratu equation.
4. To apply the Adomian Decomposition Method to the Bratu equation.

Preliminaries

Bratu Equation is a nonlinear second-order differential equation of the form

$$u'' + \lambda e^u = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

Adomian Polynomials Let the nonlinear operator F(u) and the solution expansion

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Then the Adomian polynomials A_n are defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\xi^n}\left[F\left(\sum_{k=0}^{\infty}\xi^k u_k\right)\right]_{\xi=0}, \qquad n = 0, 1, 2, \ldots$$

Conclusion

This study applied the Adomian Decomposition Method (ADM) to obtain approximate analytical solutions of Bratu-type equations. The results show that ADM generates rapidly convergent series solutions for both boundary value and initial value problems.

References

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Main Results

Algorithm of ADM

1. Rewrite the differential equation in operator form.
2. Apply the inverse operator L^{-1}
3. Decompose u(x) and nonlinear term into series.
4. Compute $u_0 + u_1 + u_2 + \cdots$ recursively.

Case 1: Boundary Value Problem

$$u'' - \pi^2 e^u = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

Let $L = \dfrac{d^2}{dx^2}$ and $Lu = \pi^2 e^u$

Step 1 : Apply inverse operator $L^{-1}(\cdot) = \int_0^x \int_0^s (\cdot)\, dt\, ds$ gives

$$u(x) = ax + \pi^2 L^{-1}(e^u)$$

Step 2 : Series representation

Assume the solution as $u(x) = \sum_{n=0}^{\infty} u_n(x)$

and the nonlinear term $F(u) = e^u = \sum_{n=0}^{\infty} A_n$

where A_n are **Adomian polynomials**

we get

$$A_0 = e^{u_0} = e^{ax}$$
$$A_1 = u_1\, e^{ax}$$
$$A_2 = \left(u_2 + \tfrac{1}{2}u_1^2\right)e^{ax}$$

Step 3 : Recursive relation

$$u_0(x) = ax$$
$$u_{n+1}(x) = \pi^2 L^{-1}(A_n), \qquad n \geq 0$$

Thus, the approximate analytical solution is

$$u(x) \approx ax + \frac{\pi^2}{a^2}(e^{ax} - ax - 1) + \frac{\pi^4}{4a^4}(e^{2ax} - 4axe^{ax} + 4e^{ax} - 2ax - 5) + \frac{\pi^6}{12a^6}\left[e^{3ax} + 6e^{2ax}(1 - ax) + 3e^{ax}(2a^2x^2 - 6ax + 5) - 6ax - 22\right] + \cdots$$

Case 2 : Boundary Value Problem

$$u'' + \pi^2 e^{-u} = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

Thus, the approximate analytical solution is

$$u(x) \approx ax - \frac{\pi^2}{a^2}(e^{-ax} + ax - 1) - \frac{\pi^4}{4a^4}\left(e^{-2ax} + 4ax\, e^{-ax} + 4e^{-ax} + 2ax - 5\right) - \frac{\pi^6}{12a^6}\left(e^{-3ax} + 6e^{-2ax}(1 + ax) + 3e^{-ax}(2a^2x^2 + 6ax - 5) + 6ax - 22\right) + \cdots$$

Case 3 : Initial Value Problem

$$u'' - 2e^u = 0, \qquad 0 < x < 1 \qquad u(0) = u'(0) = 0$$

Thus, the approximate solution is expressed as the series

$$u(x) \approx x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \cdots$$