Filling boxes in a Box

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Abstract

The Three-Dimensional Packing Problem refers to the arrangement of objects in such a way that they completely fill the interior space of a larger container without overlapping to achieve maximal space utilization. However, analyzing and enumerating all possible packing configurations in three-dimensional cases remains highly complex and requires rigorous mathematical principles for proper formulation and explanation. This research aims to investigate the mathematical foundations underlying arrangement and packing in three-dimensional space. In particular, the study analyzes and determines the number of possible ways to pack $1 \times 1 \times 1$ and $1 \times 1 \times n$ boxes into an $n \times n \times n$ container so that the space is completely filled under the specified constraints. All feasible configurations are examined under the condition that no overlapping of objects is allowed. The results provide insight into the combinatorial structure of the problem and present a systematic approach for counting packing arrangements. Furthermore, the results may be applied to storage optimization and spatial efficiency problems in broader contexts.

Notation Structure of the Box

Let the coordinates of the box be denoted by (x, y, z) where

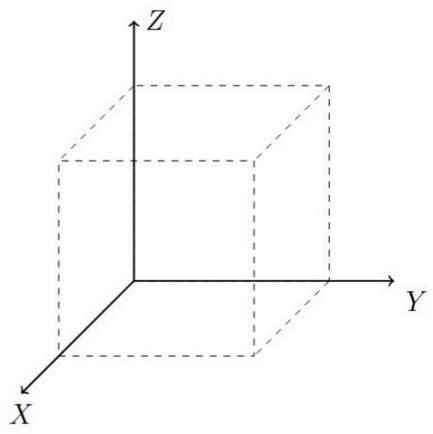
- the x-axis represents the depth
- the y-axis represents the width
- the z-axis represents the height

Consider a box of size $n \times n \times n$

The positions in the box are given by $[n] \times [n] \times [n]$

where

- the x-axis represents the depth
- the y-axis represents the width
- the z-axis represents the height



Preliminaries

Ordinary Generating Function

Let $A(x)$ be the ordinary generating function of the sequence $\{a_n\}$ defined by

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Rotational Symmetry

Rotational symmetry refers to the property of an object that remains unchanged after being rotated around a fixed axis. For a box, rotations may permute positions such as faces, vertices, or edges while preserving the overall structure.

Recurrence Relation

A recurrence relation expresses the value of a_n in terms of previous terms in the sequence. Examples include

$$a_n = 2a_{n-1}$$
$$a_n = a_{n-1} + a_{n-2}$$

Permutations and Cycles

Let X be the set of positions and let g be a permutation on X

$$g : X \to X$$

The action of g partitions the elements of X into **cycles**, where elements move in a circular sequence under the permutation.

Cycle Index

The **cycle index** of a rotation group G is defined by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} x_1^{c_1(g)} x_2^{c_2(g)} x_3^{c_3(g)} \cdots$$

where

- G is the rotation group
- $|G|$ is the number of elements in G
- $c_k(g)$ is the number of cycles of length k

The cycle index is used in **Pólya Enumeration** to count distinct arrangements under symmetry.

Counting Without Symmetry

consider the box from the top view. It forms a grid of size $n \times n$ .Thus, the box can be divided into vertical columns of size $1 \times 1 \times n$ with a total of n^2 columns.In each column, the objects can be arranged in two possible ways

1. Place n boxes of size $1 \times 1 \times 1$
2. Place one rod of size $1 \times 1 \times n$

1. Counting by Using Generating Functions

the generating function for the number of arrangements is

$$F(x) = \sum_{k=0}^{n^2} \binom{n^2}{k} x^k$$

From the binomial theorem, the generating function becomes $F(x) = F_x(x)F_y(x)F_z(x) = 3(1+x)^{n^2}$

The total number of arrangements is $3 \cdot 2^{n^2} - 2$

2. Counting by Using Recurrence Relations

Let b_m denote the number of ways to arrange objects in m columns.

When adding one more column, there are two possible choices

- place cubes of size $1 \times 1 \times 1$
- place a rod of size $1 \times 1 \times n$

Thus, the recurrence relation is $b_m = 2b_{m-1}$

with the initial condition $b_0 = 1$

the total number of arrangements is $3 \cdot 2^{n^2} - 2$

Counting with Symmetry Using Pólya Enumeration

The rotations of a box can be classified into four types and each rotation can be viewed as a **permutation** of the positions in the box, and the structure of the rotation can be described using **cycles**.

Type of Rotation	Axis of Rotation	Number	Cycle Structure
Identity	No Rotation	1	$x_1^{n^3}$
Through the center of opposite Faces	3 axes	9	$6x_1^n x_4^{\frac{n^3-n}{4}} + 3x_1^n x_2^{\frac{n^3-n}{2}}$
Through opposite vertices	4 axes	8	$8x_1^n x_4^{\frac{n^3-n}{4}}$
Through the midpoints of opposite edges	6 axes	6	$6x_1^n x_2^{\frac{n^3-n}{2}}$

From the structure of all rotations, the **cycle index** of the rotation group of the cube can be written as

$$Z(G) = \frac{1}{24}\left(x_1^{n^3} + 6x_1^n x_4^{\frac{n^3-n}{4}} + 3x_1^n x_2^{\frac{n^3-n}{2}} + 8x_1^n x_4^{\frac{n^3-n}{4}} + 6x_1^n x_2^{\frac{n^3-n}{2}}\right)$$

Conclusion

This study analyzes the number of possible arrangements in two cases: counting without rotational symmetry and counting with rotational symmetry. Generating functions and recurrence relations are used to determine the number of arrangements without symmetry. When rotational symmetry of the box is considered, Pólya's enumeration theorem and the cycle index polynomial of the rotation group are applied to count the distinct arrangements under rotation.