A Hybrid Proof of Cayley's Tree Formula from Prüfer Codes, Generating Functions and the Matrix-Tree Theorem

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Abstract

Cayley's formula is a fundamental theorem in graph theory that is utilized to determine the number of labeled trees on a set of n vertices. Despite the simplicity of the result n^{n-2}, there exist multiple methods to prove it.This independent research examines four methods of proof : Generating functions, Prüfer codes, The Matrix-tree theorem, and Degree distribution Method through multinomial coefficients. The research concludes that all methods produce consistent results and elucidate the same formula from different perspectives. A comparison of these methods underscores the strengths of each approach and the interconnections between analysis, arrangement, and algebra, thereby facilitating a deeper understanding of the structure of trees.

Objectives

This independent study seeks to examine and analyze Cayley's formula $T_n = n^{n-2}$ by evaluating various proof techniques.
The objectives are as follows:
1. To analyze and present comprehensive proofs of Cayley's formula utilizing four primary approaches.
2. To compare these methods in terms of their fundamental principles, advantages, and limitations, thereby elucidating the relationships among the various mathematical concepts involved in the proofs.

Result

Proof by Exponential Generating Functions

Let $T(S)$ be the exponential generating function for labeled trees

$$T(S) = \sum_{n=1}^{\infty} T_n \frac{S^n}{n!}$$

From the combinatorial structure of rooted trees

$$T(S) = Se^{T(S)}$$

Applying the Lagrange Inversion Formula gives the number of rooted labeled trees

$$n^{n-1}$$

Since each tree has n possible roots

$$T_n = \frac{n^{n-1}}{n} = n^{n-2}$$

Result

Proof by Prüfer Codes

Consider a labeled tree with vertices

$$\{1, 2, \ldots, n\}$$

The Prüfer sequence is constructed by repeatedly removing the leaf with the smallest label and recording the label of its adjacent vertex.
This procedure continues until only two vertices remain, producing a sequence

$$(a_1, a_2, \ldots, a_{n-2})$$

Each labeled tree corresponds uniquely to a Prüfer sequence, and conversely each Prüfer sequence determines a unique labeled tree.
Since each entry of the sequence can take any value from $\{1, 2, \ldots, n\}$

$$|\text{ Prüfer sequence }| = n^{n-2}$$

Hence

$$T_n = n^{n-2}$$

Result

Proof by Matrix-tree theorem

Consider the complete graph K_n
Its Laplacian matrix is

$$L = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}$$

$$L^* = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

By the Matrix–tree Theorem, the number of spanning trees equals the determinant of any cofactor of L

$$\tau(K_n) = det(L^*)$$

Evaluating this determinant yields $T_n = n^{n-2}$

Result

Proof by Degree Distribution Method

Let $(d_1, d_2, \ldots, d_n)$
be the degree sequence of a labeled tree.
Since a tree with n vertices has $n-1$ edges,

$$\sum_{i=1}^{n} d_i = n - 2.$$

The number of labeled trees with this degree sequence is

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$$

Summing over all valid degree sequences gives

$$T_n = n^{n-2}$$

Result

Table: Comparison of different proof methods for Cayley's Formula.

Method	Approach	Advantage	Limitations
1.Exponential Generating Functions	Analytic approach: Uses power series and functional equations such as $T(S) = Se^{T(S)}$ together with analytic tools (e.g., Lagrange inversion).	• Effective for solving counting problems involving recursively defined structures. • Strong theoretical connection with classical analytic combinatorics methods.	• Requires strong mathematical background, especially in analysis and Lagrange inversion. • Conceptually more difficult than other approaches.
2. Prüfer Codes	Combinatorial approach: Represents a labeled tree by a sequence of length $n-2$ obtained by repeatedly removing the smallest labeled leaf.	• Intuitive explanation of why the number of labeled trees is n^{n-2} • Establishes a clear bijection between trees and Prüfer sequences.	•The procedure involves several steps and may be confusing for beginners, especially for large graphs. • Not easily generalized to other combinatorial structures.
3. Matrix–tree Theorem	Algebraic approach: Uses the Laplacian matrix of a graph and computes determinants to count spanning trees.	• Very general: can count spanning trees of any graph, not only K_n • Provides precise results using linear algebra techniques.	• Determinant computation can be difficult for large matrices. • Requires knowledge of linear algebra.
4. Degree Distribution Method	Combinatorial & distribution approach: Counts trees by fixing the degree sequence and applying multinomial coefficients.	• Provides detailed insight into the number of trees with a given degree sequence $(d_1, d_2, \ldots, d_n)$ • Shows how the formula n^{n-2} arises from degree counting.	• Difficult to separate and analyze all possible degree sequences when n is large. • Requires multinomial coefficients and summation over constraints.

Conclusion

In the examination of Cayley's formula proof, $T_n = n^{n-2}$, through four primary methodologies–analytic, combinatorial, algebraic, and enumeration of degree patterns–the conclusions can be articulated as follows:

- For those seeking a detailed structural comprehension, the Prüfer code Method delivers the most transparent elucidation.
- When applying advanced tools to graph types that extend beyond complete graphs, the Matrix-tree Theorem method emerges as the most proficient.
- The Generating Function method, along with degree enumeration, offers significant insights into the intricate relationships within numerical structures and sequences.

Reference

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