ENDPOINT THEOREMS FOR GENERALIZED G-NONEXPANSIVE MAPPINGS IN HADAMARD SPACES WITH GRAPHS

AUTHOR : RAKSAYAM THONGSAMSIPCHET STUDENT ID: 650510586
ADVISOR: PROF. DR. BANCHA PANYANAK
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHAINGMAI UNIVERSITY

FACULTY OF SCIENCE
CHIANG MAI UNIVERSITY

ABSTRACT

In this study, we introduce the concept of generalized Berinde G-nonexpansive mappings in Hadamard spaces endowed with graphs and obtain an endpoint theorem for such kind of mappings. Moreover, we also obtain a convergence theorem of the Ishikawa iteration for generalized Berinde G-nonexpansive mappings in Hadamard spaces endowed with graphs.

OBJECTIVE

1. To study the existence of endpoints for generalized Berinde G-nonexpansive mappings in Hadamard spaces with graphs.
2. To study the convergence of the sequence generated by the Ishikawa iteration to an endpoint of generalized Berinde G-nonexpansive mappings in Hadamard spaces with graphs.

PRELIMINARIES

Let G be a directed graph with a set of vertices $V(G)$ and a set of edges $E(G)$. In this study, we assume that G contains all loops. Let $x, y \in V(G)$. We say that x *dominates* y if $(x, y) \in E(G)$. Let A and B be nonempty subsets of $V(G)$. We say that A *dominates* B if $(a, b) \in E(G)$ for all $a \in A$ and $b \in B$.

Let (X, d) be a metric space, C a nonempty subset of X and $G = (V(G), E(G))$ a directed graph such that $V(G) \subseteq C$. We denote by $CB(C)$ the family of nonempty closed bounded subsets of C and by $K(C)$ the family of nonempty compact subsets of C. The distance from a point x in X to a nonempty subset B of X is defined by

$$\text{dist}(x, B) := \inf\{d(x, b) : b \in B\}.$$

The *radius* of B relative to x is defined by

$$R(x, B) := \sup\{d(x, y) : y \in B\}.$$

The *Pompeiu-Hausdorff distance* on $CB(C)$ is defined by

$$H(A, B) := \max\left\{\sup_{a\in A} \text{dist}(a, B), \sup_{b\in B} \text{dist}(b, A)\right\} \text{ for all } A, B \in CB(C).$$

A multi-valued mapping $T : C \to CB(C)$ is said to be *edge-preserving* if for each $(x, y) \in E(G)$, the following implication holds:

$$u \in T(x), v \in T(y) \implies (u, v) \in E(G).$$

Definition 2.1. Let C a nonempty subset of X and $G = (V(G), E(G))$ a directed graph such that $V(G) \subseteq C$. A multi-valued mapping $T : C \to CB(C)$ is said to be *generalized Berinde G-nonexpansive* if T is edge-preserving and there exists $L \geq 0$ such that

$$H(T(x), T(y)) \leq d(x, y) + L \cdot R(x, T(x)) \text{ for all } (x, y) \in E(G).$$

Definition 2.5. A *Hadamard space* is a complete geodesic space X which satisfies the *(CN) inequality* of Bruhat and Tits:

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y), \tag{1}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Let C be a nonempty subset of X and $\{x_n\}$ be a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) := \inf\{\limsup_{n\to\infty} d(x_n, x) : x \in C\}.$$

The *asymptotic center* of x_n relative to C is the set

$$A(C, \{x_n\}) := \{x \in C : \limsup_{n\to\infty} d(x_n, x) = r(C, \{x_n\})\}.$$

Definition 2.6. Let C be a nonempty closed convex subset of X and $x \in C$. Let $\{x_n\}$ be a bounded sequence in X. We say that $\{x_n\}$ $\Delta-$*converges* to x if $A(C, \{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$.

Definition 2.8. Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a directed graph such that $V(G) \subseteq C$. Then C is said to have *property G* if for any sequence $\{x_n\}$ in C such that $\Delta - \lim_{n\to\infty} x_n = x \in C$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Let C be a nonempty subset of (X, d) and $T : C \to K(C)$ be a multi-valued mapping. A point x in C is called an *endpoint* of T if $T(x) = \{x\}$. We denote by $End(T)$ the set of all endpoints of T. A sequence $\{x_n\}$ in C is called an *approximate endpoint sequence* for T if $\lim_{n\to\infty} R(x_n, T(x_n)) = 0$.

Let C be a nonempty convex subset of X, and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$, and $T : C \to K(C)$ be a multi-valued mapping. The sequence of *Ishikawa iteration* is defined by $x_1 \in C$,

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \ n \in \mathbb{N}, \tag{2}$$

where $z_n \in T(x_n)$ such that $d(x_n, z_n) = R(y_n, T(x_n))$, and

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n z_n', \ n \in \mathbb{N} \tag{3}$$

where $z_n' \in T(y_n)$ such that $d(y_n, z_n') = R(x_n, T(y_n))$.

MAIN RESULTS

Theorem 3.1.2 *Let C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) \subseteq C$ and C has property G. Let $T : C \to K(C)$ be a generalized Berinde G-nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Theorem 3.2.5 *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to K(C)$ a generalized Berinde G-nonexpansive mapping. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of Ishikawa iteration defined by (3) such that x_1 dominates $End(T)$. Then $\{x_n\}$ $\Delta-converges$ to an endpoint of T.*

Theorem 3.2.8 *Let C be a nonempty convex subset of X. Let $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$ and C has property G. Let $T : C \to K(C)$ be a generalized Berinde G-nonexpansive mapping such that $End(T) \neq \varnothing$. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (3) such that x_1 dominates $End(T)$. If T is semicompact, then $\{x_n\}$ converges strongly to an endpoint of T.*

CONCLUSION

In conclusion, the integration of graph theory into the study of Endpoints within Hadamard spaces provides a more generalized framework than traditional metric spaces. The results obtained in this study provide important insights into the conditions required for the existence and approximation of endpoints for a broad class of multi-valued mappings.

REFERENCES

[1] A. Kaewkhao, C. Klangpraphan, B. Panyanak, *Fixed points of Osilike-Berinde-G-nonexpansive mappings in metric spaces endowed with graphs*, Carpathian J. Math., **37**:2 (2021), 311-323.

[2] C. Klangpraphan, B. Panyanak, *Fixed Point Theorems for Some Generalized Multi-Valued Nonexpansive Mappings in Hadamard Spaces*, Thai J. Math., **17**:2 (2019), 543-555.

[3] B. Panyanak, *The viscosity approximation method for multivalued G-nonexpansive mappings in Hadamard spaces endowed with graphs*, J. Fixed Point Theory Appl., **22**:90 (2020), 1-18.

[4] B. Panyanak, *Endpoints of generalized Berinde nonexpansive mappings in hyperbolic spaces*, Fixed Point Theory, **24**:1 (2023), 309-322.