A metric space of closed intervals

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Abstract

In this independent study, we have a motivation from the study of Hsien-Chung Wu on the topic "A New Concept of Fixed Point in Metric and Normed Interval Spaces". In the work, the author introduced the concept of a near fixed point and developed several types of near fixed point theorems for the set of all closed and bounded intervals in the real numbers. The idea of a null set was used to interpret positive inverse elements in this set of intervals. Based on this concept, metric interval spaces and normed interval spaces were presented, which are not metric and normed spaces in the usual sense. In this work, we construct a distance function on the equivalence classes of closed intervals in the real numbers. As a result, we obtain a metric space and prove that it satisfies the basic properties of a metric space. Moreover, we show that the set of equivalence classes of closed intervals in the real numbers, together with newly defined addition and scalar multiplication, forms a vector space.

Objective

1. To define a distance function on the metric space consisting of closed intervals of real numbers.
2. To define newly addition and scalar mutiplication that from a vector space the properties of the metric space constructed from the aforementioned definition.

Results

Let $\mathcal{I}$ be the set of all closed intervals in $\mathbb{R}$, where a closed interval $[a,b]$ is defined by

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

where $a, b \in \mathbb{R}$ with $a \leq b$.

Definition 1 The addition is given by

$$[a,b] \oplus [c,d] = [a+c, b+d].$$

Definition 2 The addition is given by

$$k[a,b] = \begin{cases} [ka, kb], & \text{if } k \geq 0, \\ [kb, ka], & \text{if } k < 0. \end{cases}$$

Example 1. $[a,b] \oplus -[a,b]$ is not equal to $[0,0]$ unless $a = b$

$$[1,4] \oplus -[1,4] = [1,4] \oplus [-4,-1] = [-3,3] \neq [0,0].$$

It is not the identity element; that is, the additional inverse of $[a,b]$ does not exist. We therefore define the null set as follows.

Definition 3

$$\Omega = \{[-k,k] : k \geq 0\}.$$

Definition 4

$[a,b] \sim [c,d]$ if and only if $[a,b] \oplus \omega_1 = [c,d] \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$.

It is clear to see that $[a,b] = [c,d]$ implies $[a,b] \sim [c,d]$ by taking $\omega_1 = \omega_2 = [0,0]$.

Theorem 5

The binary relation ~ defines an equivalence relation where $[a,b] \in \mathcal{I}$. The equivalence class of $[a,b]$ is defined as

$$\overline{[a,b]} = \{\, [c,d] \in \mathcal{I} : [a,b] \sim [c,d] \,\}$$

The collection of all equivalence classes $\overline{[a,b]}$ with $[a,b] \in \mathcal{I}$ is denoted by $\overline{\mathcal{I}}$

Example 2. $\overline{[1,3]} = \{[1-k, 3+k] : k \geq -1\}$

$\overline{[0,0]} = \{[-k,k] : k \geq 0\}$

Theorem 6 Define $\bar{d} : \overline{\mathcal{I}} \times \overline{\mathcal{I}} \to [0,\infty]$ by

$$\bar{d}(\overline{[a,b]}, \overline{[c,d]}) = |(a+b) - (c+d)|$$

Therefore, $\bar{d}$ is a metric on $\overline{\mathcal{I}}$

Example 3. $\bar{d}(\overline{[-2,4]}, \overline{[0,2]}) = |(-2+4) - (0+2)| = 0$

This can be proved by

1. We show that $\bar{d}$ is well-defined
2. We show that $\bar{d}(\overline{[a,b]}, \overline{[c,d]}) \geq 0$
3. We show that $\bar{d}(\overline{[a,b]}, \overline{[c,d]}) = 0$ if and only if $\overline{[a,b]} = \overline{[c,d]}$
4. We show that $\bar{d}(\overline{[a,b]}, \overline{[c,d]}) = \bar{d}(\overline{[c,d]}, \overline{[a,b]})$
5. We show that $\bar{d}(\overline{[a,b]}, \overline{[c,d]}) \leq \bar{d}(\overline{[a,b]}, \overline{[e,f]}) + \bar{d}(\overline{[e,f]}, \overline{[c,d]})$.

We define addition and scalar multiplication on $\overline{\mathcal{I}}$ as follows:

1. $\overline{[a,b]} \oplus \overline{[c,d]} = \overline{[a+c, b+d]}$
2. $k\overline{[a,b]} = \begin{cases} \overline{[ka,kb]}, & \text{if } k \geq 0, \\ \overline{[kb,ka]}, & \text{if } k < 0. \end{cases}$

Example 4. $3\overline{[2,5]} = \overline{[3\times 2, 3\times 5]} = \overline{[6,15]}$

$-5\overline{[4,6]} = \overline{[-5\times 6, -5\times 4]} = \overline{[-30,-20]}$

The set $\overline{\mathcal{I}}$, together with the binary operation $\oplus$ and scalar multiplication defined as above, constitutes a vector space. The verification is based on the following properties.

1. $v \oplus w = w \oplus v$
2. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$
3. $v \oplus \mathbf{0} = v$
4. $v \oplus (-v) = 0$
5. $\alpha(v \oplus w) = \alpha v \oplus \alpha w$
6. $(\alpha \oplus \beta)v = \alpha v \oplus \beta v$
7. $(\alpha\beta)v = \alpha(\beta v)$
8. $1v = v$

Conclusion

The results of this study clearly demonstrate that although the set of closed intervals in the real numbers cannot form a vector space directly under the usual operations, an appropriate vector space structure can be constructed by considering suitable equivalence relations and equivalence classes. The resulting structure satisfies all the axioms of a vector space, including the existence of a zero vector, additive inverses, and the distributive property of scalar multiplication.

Reference

Hsien-Chung Wu, A New Concept of Fixed Point in Metric and Normed Interval Spaces, Mathematics, 2018; vol 6(11): page 219