SOME BASIC PROPERTIES OF HYPERBOLIC LEONARDO AND HYPERBOLIC FRANCOIS OCTONIONS

Author : Chanidapha khrueangkham Advisor : Assoc. Prof. Dr. Narawadee Phudolsitthiphat
Department of Mathematics, Faculty of Science, Chiang Mai University

Abstract

Octonions form an eight-dimensional vector. In this work, we extend the theory of hypercomplex numbers by introducing two new types of octonions, namely hyperbolic Leonardo octonions and hyperbolic Francois octonions. The main objective of this independent study is to investigate and establish the fundamental algebraic properties of these newly defined structures. In particular, we derive Binet-type formulas, which enable the direct computation of the n-th term without relying on recursive processes. Furthermore, explicit formulas for conjugates and norms are obtained.

Objectives

1. To study the definition and structure of hyperbolic Leonardo and hyperbolic Francois octonions.
2. To prove some basic properties of hyperbolic Leonardo and hyperbolic Francois octonions, such as the Binet's formulas, conjugate and norm.

Preliminaries

Definition 2.3.1 [4] Let $n \geq 0$ be an integer. The Leonardo sequence $\{L_n\}$ is defined by recurrence

$$L_0 = 1,\ L_1 = 1,\ \text{ and }\ L_{n+2} = L_{n+1} + L_n + 1$$

where L_n is the n-th Leonardo number.

An expression of the relationship between Leonardo and Fibonacci numbers is

$$L_n = 2\mathcal{F}_{n+1} - 1,\ n \geq 0.$$

First few terms of these sequences are

$$1,\ 1,\ 3,\ 5,\ 9,\ 15,\ 25,\ 41,\ 67,\ 109, \ldots$$

Theorem 2.4.1 [4] Let $n \geq 0$ be an integer. The Binet formulas for the Leonardo sequence is defined by

$$L_n = 2\Big(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\Big) - 1,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Definition 2.5.1 [4] Let $n \geq 2$ be an integer. The Francois sequence $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2} + 1,$$

with initial conditions $F_0 = 2$ and $F_1 = 1$. Here, F_n is the n-th Francois number. First few terms of this sequence are,

$$2,\ 1,\ 4,\ 6,\ 11,\ 18,\ 30,\ 49,\ 80,\ 130,\ 211.$$

The recurrence can also be written as follows

$$F_{n+3} = 2F_{n+2} - F_n.$$

Theorem 2.6.1 [4] Let $n \geq 0$ be an integer. The Binet formulas for the Francois sequence is defined by

$$F_n = \alpha^n + \beta^n + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Definition 2.12.1 [7] A hyperbolic octonion is an element of the form

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 + q_4 e_4 + q_5 e_5 + q_6 e_6 + q_7 e_7,$$

where $q_0,\ q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6,\ q_7$ are real numbers, and i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4^2 = 1)$ is a counter imaginary unit. The bases of octonions are satisfying the following properties:

$$i_1 e_4 = e_5,\ \ i_2 e_4 = e_6,\ \ i_3 e_4 = e_7,\ \ e_4^2 = e_5^2 = e_6^2 = e_7^2 = 1.$$

Moreover, for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = kq_0 + kq_1 i_1 + kq_2 i_2 + kq_3 i_3 + kq_4 e_4 + kq_5 e_5 + kq_6 e_6 + kq_7 e_7.$$

and conjugate and norm of the hyperbolic octonions are

$$\overline{q} = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3 - q_4 e_4 - q_5 e_5 - q_6 e_6 - q_7 e_7.$$

and

$$\|q\| = \sqrt{|q\overline{q}|} = \sqrt{|q_0^2 + q_1^2 + q_2^2 + q_3^2 - q_4^2 - q_5^2 - q_6^2 - q_7^2|}.$$

Results

Definition 3.1.1 Let $n \geq 0$ be an integer. The hyperbolic Leonardo octonion $\{OL_n\}$ is defined by

$$OL_n = L_n + L_{n+1} i_1 + L_{n+2} i_2 + L_{n+3} i_3 + L_{n+4} e_4 + L_{n+5} e_5 + L_{n+6} e_6 + L_{n+7} e_7$$

where L_n is the n-th Leonardo number and i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4^2 = 1)$ is a counter imaginary unit. The bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5,\ \ i_2 e_4 = e_6,\ \ i_3 e_4 = e_7,\ \ e_4^2 = e_5^2 = e_6^2 = e_7^2 = 1.$$

Definition 3.1.3 Let $n \geq 0$ be an integer. Then $\{\overline{OL_n}\}$ is defined by

$$\overline{OL_n} = L_n - L_{n+1} i_1 - L_{n+2} i_2 - L_{n+3} i_3 - L_{n+4} e_4 - L_{n+5} e_5 - L_{n+6} e_6 - L_{n+7} e_7$$

where L_n is the n-th Leonardo number and i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4 = 1)$ is a counter imaginary unit. The bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5,\ \ i_2 e_4 = e_6,\ \ i_3 e_4 = e_7,\ \ e_4^2 = e_5^2 = e_6^2 = e_7^2 = 1.$$

Theorem 3.2.1 Let $n \geq 0$ be an integer. Then the following are true

1. $OL_n + \overline{OL_n} = 2L_n$
2. $OL_n - \overline{OL_n} = 2OL_n - 2L_n$
3. $OL_n(\overline{OL_n}) = \sum_{k=n}^{n+3} L_k^2 - \sum_{k=n+4}^{n+7} L_k^2.$

Theorem 3.2.5 Norm of hyperbolic Leonardo octonions $\{\|OL_n\|\}$ is

$$\|OL_n\| = \sqrt{|OL_n\overline{OL_n}|} = \sqrt{|L_n^2 + L_{n+1}^2 + L_{n+2}^2 + L_{n+3}^2 - L_{n+4}^2 - L_{n+5}^2 - L_{n+6}^2 - L_{n+7}^2|}.$$

Theorem 3.3.1 Let $n \geq 0$ be and integer. We have

$$OL_n = 2\Big(\frac{\widehat{\alpha}\alpha^{n+1} - \widehat{\beta}\beta^{n+1}}{\alpha - \beta}\Big) - \widehat{1}$$

where $\widehat{\alpha} = 1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3 + \alpha^4 e_4 + \alpha^5 e_5 + \alpha^6 e_6 + \alpha^7 e_7$
$\widehat{\beta} = 1 + \beta i_1 + \beta^2 i_2 + \beta^3 i_3 + \beta^4 e_4 + \beta^5 e_5 + \beta^6 e_6 + \beta^7 e_7$
$\widehat{1} = 1 + i_1 + i_2 + i_3 + e_4 + e_5 + e_6 + e_7.$

Definition 3.4.1 Let $n \geq 0$ be an integer. The hyperbolic Francois octonions $\{OF_n\}$ is defined by

$$OF_n = F_n + F_{n+1} i_1 + F_{n+2} i_2 + F_{n+3} i_3 + F_{n+4} e_4 + F_{n+5} e_5 + F_{n+6} e_6 + F_{n+7} e_7$$

where F_n is the n-th Francois number and i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4^2 = 1)$ is a counter imaginary unit. The bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5,\ \ i_2 e_4 = e_6,\ \ i_3 e_4 = e_7,\ \ e_4^2 = e_5^2 = e_6^2 = e_7^2 = 1.$$

Definition 3.4.3 Let $n \geq 0$ be an integer. Then $\{\overline{OF_n}\}$ is defined by

$$\overline{OF_n} = F_n - F_{n+1} i_1 - F_{n+2} i_2 - F_{n+3} i_3 - F_{n+4} e_4 - F_{n+5} e_5 - F_{n+6} e_6 - F_{n+7} e_7$$

where F_n is the n-th Francois number and i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4 = 1)$ is a counter imaginary unit. The bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5,\ \ i_2 e_4 = e_6,\ \ i_3 e_4 = e_7,\ \ e_4^2 = e_5^2 = e_6^2 = e_7^2 = 1.$$

Theorem 3.5.1 Let $n \geq 0$ be an integer. Then the following are true:

1. $OF_n + \overline{OF_n} = 2F_n,$
2. $OF_n - \overline{OF_n} = 2OF_n - 2F_n,$
3. $OF_n(\overline{OF_n}) = \sum_{k=n}^{n+3} F_k^2 - \sum_{k=n+4}^{n+7} F_k^2.$

Theorem 3.5.5 Norm of hyperbolic Francois octonions $\{\|OF_n\|\}$ is

$$\|OF_n\| = \sqrt{|OF_n\overline{OF_n}|} = \sqrt{|F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 - F_{n+4}^2 - F_{n+5}^2 - F_{n+6}^2 - F_{n+7}^2|}.$$

Theorem 3.6.1 Let $n \geq 0$ be and integer. We have

$$OF_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \Big(\frac{\widehat{\alpha}\alpha^{n+1} - \widehat{\beta}\beta^{n+1}}{\alpha - \beta}\Big) - \widehat{1}$$

where $\widehat{\alpha} = 1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3 + \alpha^4 e_4 + \alpha^5 e_5 + \alpha^6 e_6 + \alpha^7 e_7$
$\widehat{\beta} = 1 + \beta i_1 + \beta^2 i_2 + \beta^3 i_3 + \beta^4 e_4 + \beta^5 e_5 + \beta^6 e_6 + \beta^7 e_7$
$\widehat{1} = 1 + i_1 + i_2 + i_3 + e_4 + e_5 + e_6 + e_7.$

Conclusion

In this work, we study the definition and structure of hyperbolic Leonardo and hyperbolic Francois octonions.
In addition, we prove some basic properties , such as the Binet's formulas, conjugate and norm.

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