Arbitrage portfolios of arbitrary investing

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Abstract

This independent study explores investment strategies across diverse scenarios by constructing a payoff matrix derived from rates of return. The primary focus is the analysis of risk-free portfolios, specifically arbitrage portfolios, which prioritize profit generation by leveraging institutional credit instead of utilizing personal equity. Subsequently, Linear Programming (LP) is employed as a robust analytical framework to detect the presence of arbitrage. By formulating sophisticated linear programming models, we extend our analysis to identify portfolios capable of generating consistent profits across all possible states. Ultimately, this study constructs profitable investment portfolios that align with the practical financial constraints of real-world environments. Our findings conclude that optimal portfolio selection is intrinsically linked to the interplay between rates of return and borrowing costs. In this regard, linear programming serves as a powerful instrument, empowering investors to determine the most efficient asset allocation and optimal leverage.

objectives

- To investigate the correlation between return matrices and investment portfolios.
- To determine the maximum return of an arbitrage portfolio using optimization techniques.

Preliminaries

Return matrix

For an investment in m assets under n possible states, the matrix $R = (r_{ij})$ of size $m \times n$ is called the **return matrix** if r_{ij} represents the total return including the principal from investing in i asset in state j, given an initial investment of 1 baht.

Portfolio

For an investment in m assets, we define the matrix

$$Y = [y_1 \quad y_2 \quad \cdots \quad y_m]$$

where y_i is the amount invested in asset i.

Investment Return

We refer to YR as the returns from the investment, where Y is the investment portfolio and R is the return matrix defined by

$$Y = [y_1\ y_2\ \cdots\ y_m] \quad \text{and} \quad R = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{bmatrix}.$$

The resulting returns from the investment are:

$$YR = [y_1\ y_2\ \cdots\ y_m] \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{bmatrix} = [a_1y_1 + a_2y_2 + \cdots + a_my_m \quad b_1y_1 + b_2y_2 + \cdots + b_my_m]$$

Arbitrage Portfolio

We call an investment portfolio $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]$ such that $y_1 + y_2 + \cdots + y_m = 0$ with the return matrix R an arbitrage portfolio, if no component of YR is less than o and at least one component is strictly more than o.

Methodology

- Reviewing fundamental matrix theories and investment analysis concepts.
- Utilizing augmented matrices to test the conventional theorem regarding state prices and find counterexamples.
- Formulating Linear Programming (LP) models to detect and maximize arbitrage opportunities.

LP Methodology

$$R = \begin{bmatrix} 1.20 & 1.05 & 1.25 & 1.15 \\ 1.08 & 1.30 & 1.10 & 1.20 \\ 1.06 & 1.06 & 1.06 & 1.06 \\ 1.04 & 1.08 & 1.05 & 1.07 \end{bmatrix}$$

Constraints :

1. $1.20y_1 + 1.08y_2 - 1.06y_3 - 1.04y_4 \geqslant 0$	Return in Scenario 1
2. $1.05y_1 + 1.30y_2 - 1.06y_3 - 1.08y_4 \geqslant 0$	Return in Scenario 2
3. $1.25y_1 + 1.10y_2 - 1.06y_3 - 1.05y_4 \geqslant 0$	Return in Scenario 3
4. $1.15y_1 + 1.20y_2 - 1.06y_3 - 1.07y_4 \geqslant 0$	Return in Scenario 4
5. $y_3 + y_4 \leqslant 10000$	Given a loan amount of 10,000 units.
6. $(y_3 + y_4) - (y_1 + y_2) \geqslant E$	Operating Labor Cost
7. $y_1, y_2 \geqslant 0$	The investment
8. $y_3, y_4 \geqslant 0$	loan amount
9. $E \geqslant 0$	remaining balance must be positive

Objective Functions :

- $\max Z = \sum_{i=1}^{n} S_i$ ➡ $\max Z = 4400$ Linear programming can identify a portfolio that generates a maximum total return of 4,400 units.
- $\max Z = E$ ➡ $\max Z \approx 854.46$ The optimal cash reserve (Max Z) is approximately 854.46 units.
- $\max Z = P$, $E = 0$ ➡ $\max Z \approx 854.46$ The guaranteed minimum profit (Max Z) is approximately 983.78 units.
- $\max Z = P$, $E = 5\%$ ➡ $\max Z \approx 408.11$ The guaranteed minimum profit (Max Z) is approximately 408.11 units.

Results

- LP for Arbitrage Detection: Linear Programming effectively detects arbitrage. An objective function maximizing returns (Z) guarantees the existence of an arbitrage portfolio if $Z > 0$.
- Real-World Application: Successfully applied LP to real-world financial data (dividend stocks and bank loan rates) to construct a profitable portfolio with zero initial personal equity.
- Weighted Scenarios: Implemented weighted objective functions (Weighted l_1-norm) to handle uncertainties, such as political scenarios, optimizing decisions across multiple possible outcomes.

Conclusion

Linear Programming serves as a robust and highly efficient analytical framework, empowering investors to determine optimal asset allocation and leverage under practical financial constraints.

Reference

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