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Contributed Paper

Associated Prime Submodules and Related Results

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ABSTRACT

The notions of associated and supported prime submodules, as a generalization of associated and supported prime ideals, have been introduced by Divaani-Aazar and Esmkhani. In this paper, we will try to develop this theory and give some relations between associated, supported, maximal and prime submodules.

Keywords: associated prime submodules, prime submodules, quasi multiplication modules.

1. INTRODUCTION

Throughout this paper R is a commutative ring with identity and all modules are unitary. Prime submodules have been studied by many authors. A submodule N of an R -module M is said to be prime (or p -prime) if $rx \in N$ for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in p = (N : M)$. Various papers on prime submodules are devoted to the generalization of related results of prime ideals to prime submodules, see for example [1]. In a recent paper of this case, Divaani-Aazar and Esmkhani [2] introduced the notion of associated and supported prime submodules as a generalization of associated and supported prime ideals and extended some of the most important results concerning associated prime ideals. For instance, they proved that for any multiplication module M over a Noetherian ring R , if $0 = \bigcap_{i=1}^n Q_i$ is a minimal primary decomposition of the zero submodule of M , then $\text{Ass}_p M = \{\text{rad}(Q_i) : i = 1, 2, \dots, n\}$. To see the main properties of the

sets of associated and supported prime submodules one is referred to [2]. Recall that an R -module M is called a multiplication module if for each submodule N of M , $N = IM$ for some ideal I of R .

Because the theory of associated primes is very important in the study of modules, it seems that it is necessary to develop the theory of associated prime submodules and this is the main goal of this article.

Let M be a finitely generated R -module and p a prime ideal of R . McCasland and Smith in [3] introduced the notion of $M(p)$ defined by,

$M(p) = \{x \in M : rx \in pM, \text{ for some } r \in R \setminus p\}$ which is a submodule of M and Divaani-Aazar with Esmkhani in [2] used this concept to generalize the notion of associated prime ideals. The submodule $M(p)$ is well behaved when we need to construct prime submodules. For example, it is easy to see that if M is finitely generated and p contains the annihilator

of M , then $M(p)$ is a p -prime submodule of M . The reader can find additional information about this notion in [2, 3].

The sets of prime and maximal submodules of an R -module M are denoted by $\text{Spec } M$ and $\text{Max } M$, respectively. Also, for unexplained definitions and terminologies, we refer the reader to [2, 4, 5].

2. MAIN RESULTS

Let M be an R -module. The set of associated prime ideals of M is denoted by $\text{Ass}_R M$ and it is the set of prime ideals p such that there exists $x \in M$ with $p = \text{Ann}(x)$. Also, the support of M , written $\text{Supp}_R M$, is the set of prime ideals p such that there exists $x \in M$ with $p \supseteq \text{Ann}(x)$.

In the following we list some of the important theorems on the associated and supported prime ideals and then we generalize them.

Theorem 2.1 $\text{Ass}_R(\bigoplus_{i \in I} M_i) = \bigcup_{i \in I} \text{Ass}_R(M_i)$, if $\{M_i\}$ is a family of R -modules.

Proof. See [6, p. 67]. •

Theorem 2.2 Let M be a module over Noetherian ring R and $p \in \text{Spec } R$. Then $p \in \text{Supp}_R M$ if and only if $p \supseteq q$ for some $q \in \text{Ass}_R M$.

Proof. See [6, Proposition 4.3.3]. •

Theorem 2.3 Let M be a finitely generated module over Noetherian ring R . Then the following conditions are equivalent.

- (i) M is of finite length.
- (ii) If $p \in \text{Ass}_R M$, then p is a maximal ideal of R .
- (iii) If $p \in \text{Supp}_R M$, then p is a maximal ideal of R .

Proof. See [6, Corollary 4.3.4.2]. •

Following Divaani-Aazar and Esmkhani in [2], we define:

DEFINITION

(i) An R -module M is said to be weakly finitely generated if for each $p \in \text{Supp}_R M$, the submodule $M(p)$ of M is proper.

(ii) Let M be a weakly finitely generated R -module. The sets of associated and supported prime submodules of M are denoted by $\text{Ass}_p M$ and $\text{Supp}_p M$ respectively and defined by

$$\text{Ass}_p M = \{M(p) : p \in \text{Ass}_R M\} \text{ and}$$

$$\text{Supp}_p M = \{M(p) : p \in \text{Supp}_R M\}.$$

For any R -module M , it is known that $\text{Supp}_R M = \emptyset$ if and only if $M = 0$. But, it is not always true that $\text{Supp}_p M = \emptyset$ if and only if $M = 0$, if we remove the condition of weakly finitely generated; see the next example.

Example 2.4 Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. It is easy to check that \mathbb{Q} isn't weakly finitely generated as a \mathbb{Z} -module. Indeed, $M(p) = M$, for any $p \in \text{Supp}_{\mathbb{Z}} \mathbb{Q}$ and it means that $\text{Supp}_p \mathbb{Q} = \emptyset$, whereas $\mathbb{Q} \neq 0$.

The above example shows that the condition of weakly finitely generated in the definitions of associated and supported prime submodules, is necessary and can't be removed.

Now, we are ready to state and prove our results.

Lemma 2.5 Let $\{M_i\}_{i \in I}$ be a family of R -modules. Then $(\bigoplus_{i \in I} M_i)(p) = \bigoplus_{i \in I} M_i(p)$, for each $p \in \text{Spec } R$.

Proof. Let p be a prime ideal of R and $m \in (\bigoplus_{i \in I} M_i)(p)$. So there exists $r \in R \setminus p$ such that $rm \in p(\bigoplus_{i \in I} M_i)$. Since $m \in \bigoplus_{i \in I} M_i$, we can write $m = (m_i)$ where $m_i \in M_i$, for each $i \in I$. It is easy to check that $p(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} (pM_i)$ and so we have

$$rm = r(m_i) = (rm_i) \in p(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} (pM_i).$$

This implies that $m_i \in M_i(p)$ for each $i \in I$ and so $m \in \bigoplus_{i \in I} M_i(p)$. Hence, $(\bigoplus_{i \in I} M_i)(p) \subseteq \bigoplus_{i \in I} M_i(p)$. Now, suppose that $m = (m_i) \in \bigoplus_{i \in I} M_i(p)$ and $m_{i_1}, m_{i_2}, \dots, m_{i_n}$ are the non-zero components in M . Since $m_{i_j} \in M_{i_j}(p)$, there exist $r_{i_j} \in R \setminus p$ such that $r_{i_j} m_{i_j} \in pM_{i_j}$ for $j = 1, 2, \dots, n$. Now, set $r = r_{i_1} r_{i_2} \dots r_{i_n}$. It is clear that $r \in R \setminus p$ and so we have $rm \in \bigoplus_{i \in I} (pM_i) = p(\bigoplus_{i \in I} M_i)$. Hence $\bigoplus_{i \in I} M_i(p) \subseteq (\bigoplus_{i \in I} M_i)(p)$ and this completes the proof. •

Theorem 2.6 Let $\{M_i\}_{i \in I}$ be a family of weakly finitely generated R -modules. Then $\text{Ass}_p(\bigoplus_{i \in I} M_i) = \{\bigoplus_{i \in I} (M_i(p)) : M_i(p) \in \text{Ass}_p M_i, \text{ for some } i \in I\}$

Proof. It is easy to see that $\bigoplus_{i \in I} M_i$ is a weakly finitely generated R -module and so we have

$$\begin{aligned} \text{Ass}_p(\bigoplus_{i \in I} M_i) &= \{(\bigoplus_{i \in I} M_i)(p) : p \in \text{Ass}_R(\bigoplus_{i \in I} M_i)\} \\ &= \{\bigoplus_{i \in I} M_i(p) : p \in \bigcup_{i \in I} \text{Ass}_R(M_i)\} = \\ &= \{\bigoplus_{i \in I} M_i(p) : M_i(p) \in \text{Ass}_p(M_i), \text{ for some } i \in I\}. \end{aligned}$$

As in [2], we say that an R -module M is a quasi multiplication if $M(p) = pM$, for all $p \in \text{Supp}_R M$. Trivially, multiplication modules are quasi multiplication. Also, any flat module is quasi multiplication. One can find some relations between flat modules and multiplication modules in [7, 8].

Theorem 2.7 Let M be a quasi multiplication weakly finitely generated module over Noetherian ring R . If $p \in \text{Spec } R$ and $M(p) \in \text{Spec } M$, then $M(p) \in \text{Supp}_p M$ if and only if $M(p) \supseteq Q$ for some $Q \in \text{Ass}_p M$.

Proof. Let $M(p) \in \text{Supp}_p M$. It is clear that $(M(p) : M) = p \in \text{Supp}_R M$. Now, by Theorem 2.2 there is a $q \in \text{Ass}_R M$ such that $p \supseteq q$. Since M is a quasi multiplication module, $M(p) \supseteq M(q)$ and $M(q) \in \text{Ass}_p M$. Set $M(q) = Q$. Hence, $M(p) \supseteq Q$ for some $Q \in \text{Ass}_p M$. Conversely, assume that $M(p) \supseteq Q$ for some $Q \in \text{Ass}_p M$. Set $q = (Q : M)$, then $q \in \text{Ass}_R M$ and $M(q) = Q$ so $M(p) \supseteq M(q)$. Then $p = (M(p) : M) \supseteq (M(q) : M) = q$. Therefore, $p \in \text{Supp}_R M$ and so $M(p) \in \text{Supp}_p M$, as required. •

Theorem 2.8 Let M be a multiplication module over Noetherian ring R and $p \in \text{Spec } R$. The following conditions are equivalent:

- (i) M is of finite length.
- (ii) if $Q \in \text{Ass}_p M$, then Q is a maximal submodule of M , i.e., $\text{Ass}_p M \subseteq \text{Max } M$.
- (iii) $\text{Supp}_p M \subseteq \text{Max } M$.

Proof. It turns out by [9, Corollary 3.9], any multiplication module over Noetherian

ring is finitely generated. Now, we have:

(i) \Rightarrow (ii) See [2, Corollary 2.7].

(ii) \Rightarrow (iii) Let $P \in \text{Supp}_p M$. By Theorem 2.7, there is a $Q \in \text{Ass}_p M$ such that $P \supseteq Q$ and by (ii), Q is a maximal submodule. So $P = Q$.

(iii) \Rightarrow (i) Since M is a Noetherian multiplication module and $\text{Ass}_p M \subseteq \text{Supp}_p M \subseteq \text{Max } M$, the result follows from [2, Corollary 2.7]. •

We end this section with a question. It is well known that, if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of R -modules and R -homomorphism, then

$$(i) \text{Ass}_R L \subseteq \text{Ass}_R M \subseteq \text{Ass}_R L \cup \text{Ass}_R N,$$

$$(ii) \text{Supp}_R M = \text{Supp}_R L \cup \text{Supp}_R N.$$

Is it possible to generalize this theorem to associated and supported prime submodules?

3. COMPLEMENT TO SUPPORTED PRIME SUBMODULES

In this section, for an R -module M , we study the set of supported prime submodules and their complements to gain useful information about M . A submodule $C \subseteq M$ is said to be a complement to S in M if C is maximal with respect to the property that $C \cap S = 0$. By Zorn's lemma, any submodule S has a complement; in fact, any submodule C_0 with $C_0 \cap S = 0$ can be enlarged in to a complement to S in M . For more details about complements we refer the reader to [10]. A submodule N of an R -module M is said to be strongly irreducible if for submodules L and K of M , the inclusion $L \cap K \subseteq N$ implies that either $L \subseteq N$ or $K \subseteq N$.

The first theorem of this section gives a characterization of multiplication modules in terms of supported prime submodules and maximal submodules. To prove this fact, we need to the following lemma.

Lemma 3.1 Let M be a finitely generated R -module. Then, for any $p \in \text{Supp}_R M \cap \text{Max } R$, $M(p) = pM$ and $M(p)$ is a p -prime submodule of M .

Proof. Let $p \in \text{Supp}_R M \cap \text{Max } R$.

p -primeness of $M(p)$ and pM follows from [2, Lemma 2.2 (ii)] and [11, Corollary 3], respectively. It is clear that $pM \subseteq M(p)$. Also, by [2, Corollary 2.2 (iii)], $M(p) \subseteq pM$, and so $M(p) = pM$. •

We can now state our first theorem in this section.

Theorem 3.2 Let M be a finitely generated R -module. Then the following statements are equivalent.

- (i) M is a multiplication R -module.
- (ii) $\text{Spec } M = \text{Supp}_p M$.
- (iii) $\text{Max } M = \{M(p) \in \text{Supp}_p M : p \in \text{Max } R\}$.

Proof. (i) \Rightarrow (ii) It is enough to show $\text{Spec } M \subseteq \text{Supp}_p M$. Let $N \in \text{Spec } M$, then $N = pM$ for some prime ideal p of R with $\text{Ann } M \subseteq p$, by [9, Corollary 2.11]. Since M is a finitely generated R -module, $p \in \text{Supp}_R M$ and hence, $N \in \text{Supp}_p M$, as desired.

(ii) \Rightarrow (iii) Let $L \in \text{Max } M \subseteq \text{Spec } M = \text{Supp}_p M$. Then there exists $p \in \text{Supp}_R M$ such that $L = M(p)$. By [11, Proposition 4], the ideal $(L : M) = (M(p) : M) = p$ is a maximal ideal of R . Conversely, let $M(m) \in \text{Supp}_p M$ such that $m \in \text{Max } R$. Then there exists $K \in \text{Max } M \subseteq \text{Spec } M = \text{Supp}_p M$ such that $M(m) \subseteq K$, because M is a finitely generated R -module. So there exists $m' \in \text{Max } R$ such that $K = M(m')$. Hence, $m = m'$ and $M(m) = K \in \text{Max } M$.

(iii) \Rightarrow (i) By [12, Theorem 3.5], it is enough to show that for any $m \in \text{Max } R$, M/mM is a cyclic module. Let $m \in \text{Max } R$. If $m \in \text{Supp}_R M$, then $M(m) = mM$ is a prime submodule of M , by Lemma 3.1. Also, $M(m) \in \text{Supp}_p M$ and by (iii), $M(m) \in \text{Max } M$. Hence, M/mM is simple and so is a cyclic module. If $m \notin \text{Supp}_R M$, then it is easy to check that $M(m) = M$ and so M/mM is a cyclic module. •

As we have seen in Theorem 3.2, in the case that M is a finitely generated multiplication R -module, $\text{Spec } M = \text{Supp}_p M$ and $\text{Max } M = \{M(p) \in \text{Supp}_p M : p \in \text{Max } R\}$.

From now, we remove the condition of finitely generated and focus on the notions of strongly irreducible and complements to obtain similar relations.

Lemma 3.3 Let M be a multiplication R -module and N be a prime submodule of M . Then N is strongly irreducible.

Proof. See [13, Theorem 3.1]. •

Corollary 3.4 Let M be a multiplication R -module and N a submodule of M . If L is a complement to N in M , then either $N \subseteq pM$ or $L \subseteq pM$ for each $pM \in \text{Supp}_p M$.

Proof. It is clear that $N \cap L = 0$, hence $N \cap L \subseteq pM$ for each $pM \in \text{Supp}_p M$. Now we have either $N \subseteq pM$ or $L \subseteq pM$ for each $pM \in \text{Supp}_p M$. •

Theorem 3.5. Let M be a multiplication R -module and $pM \in \text{Supp}_p M$. Then either complement to pM is zero or pM includes all complements to other members of $\text{Supp}_p M$.

Proof. Assume that $p_i M \in \text{Supp}_p M$ and C_i is complement to $p_i M$. If $p_j M$ is another member of $\text{Supp}_p M$ with complement C_j , then $p_i M \subseteq p_j M$ or $C_i \subseteq p_j M$ and $p_j M \subseteq p_i M$ or $C_j \subseteq p_i M$. If $p_i M \subseteq p_j M$, then $p_j M \not\subseteq p_i M$ which conclude that $C_i \subseteq p_j M$. But C_i is complement to $p_i M$, hence $C_i = 0$. It is clear that if $p_j M \not\subseteq p_i M$, then $C_j \subseteq p_i M$. The case $p_i M \subseteq p_j M$ can be proved similarly. •

The proof of the next Corollary is easy, so we omit it.

Corollary 3.6 Let M be a multiplication R -module and $pM \in \text{Supp}_p M$. If pM has a non-zero complement and $|pM| < \infty$, then the number of elements of $\text{Supp}_p M$ with non-zero complement is finite.

Lemma 3.7 Let M be a multiplication R -module. If the intersection of two members of $\text{Supp}_p M$ is zero and they have non-zero complements, then they are complements to each other.

Proof. Let $p_i M$ and $p_j M$ be two members of supported prime submodules such that $p_i M \cap p_j M = 0$ and they have non-zero

complements C_i and C_j , respectively. By Corollary 3.4, $C_i \subseteq p_i M$ and $C_j \subseteq p_j M$. Since C_i and C_j are maximal with respect to the property that $C_i \cap p_i M = 0$ and $C_j \cap p_j M = 0$, it follows that $p_i M = C_i$ and $p_j M = C_j$. •

Lemma 3.8 Let M be a multiplication R -module and any member of $\text{Max } M$ has a non-zero complement. Then $\text{Spec } M = \text{Max } M$.

Proof. It is enough to show that $\text{Spec } M \subseteq \text{Max } M$. Suppose that $pM \in \text{Spec } M$. Since M is a multiplication R -module, there exists $mM \in \text{Max } M$ such that contains pM . Let C be complement to mM . Since pM is strongly irreducible, either $pM = mM$ or $C \subseteq pM$ which $C \subseteq pM$ conclude that $C = 0$ and this is contradiction to assumption. Hence, $\text{Spec } M \subseteq \text{Max } M$. •

Corollary 3.9. Let M be a multiplication module over Noetherian ring R . If any member of $\text{Max } M$ has a non-zero complement, then $\text{Supp}_p M = \text{Ass}_p M = \text{Spec } M = \text{Max } M$.

Proof. Use Theorem 3.2, Lemma 3.8, [2, Corollary 2.7 and Lemma 2.6]. •

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REFERENCES

- [1] Atani E.S. and Darani A.Y., On quasi-primary submodules, *Chiang Mai J. Sci.*, 2006; **33**(3): 249-254.
- [2] Divaani-Aazar K. and Esmkhani M.A., Associated prime submodules of finitely generated modules, *Comm. Algebra*, 2005; **33**: 4259-4266.
- [3] McCasland R.L. and Smith P.F., Prime Submodules of Noetherian modules, *Rocky Mountain J. Math.*, 1993; **23**: 1041-1062.
- [4] Bourbaki N., *Algebra commutative*, Herman, Paris, 1967.
- [5] Rotman J.J., *An introduction to homological algebra*, 1st Edn., Academic Press, New York, 1974.
- [6] Knight J.T., *Commutative Algebra*, 1st Edn., Cambridge University, London, 1971.
- [7] Ali M.M. and Smith D.J., Projective, Flat and Multiplication modules, *N. Z. J. Math.*, 2002; **31**: 115-129.
- [8] Naoum N.G., Flat modules and multiplication modules, *Period. Math. Hungar.*, 1990; **21**: 309-317.
- [9] El-Bast Z.A. and Smith P.F., Multiplication modules, *Comm. Algebra*, 1998; **16**: 755-779.
- [10] Lam T.Y., *Lectures on modules and rings*, Graduate texts in mathematics, Springer-Verlag, York, 1999; 189.
- [11] Lu C.P., Prime submodule of modules, *Comment. Math. Univ. St. Paul*, 1984; **33**: 61-69.
- [12] McCasland R.L., Moore M.E. and Smith P.F., On the spectrum of a module over Noetherian ring, *Comm. Algebra*, 1997; **25**: 79-103.
- [13] Atani E.S., Strongly irreducible submodules, *Bull. Korean Math. Soc.*, 2005; **42**: 121-131.