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Contributed Paper

Strong Coupling Solutions of Superposed Harmonic Oscillator with a Generalized Even Power Potential

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Abstract

Large coupling expansions of eigen energies and wave functions of a harmonic potential superposed with a general even power have been evaluated. These general expansions are then applied to harmonic and quartic anharmonic oscillator problems. The results are found to be in good agreement with those of earlier workers.

Keyword: Strong coupling, Harmonic oscillator, Generalized even power.

Introduction

Perturbation theory, although used for deriving first or second order approximations, was not generally considered to be giving true series solution from which at least analytic properties could be inferred. In various investigations Aly et al. [1], Müller- Kirsten et al.[2-4] and Sharma et al.[5-8] have shown that this disregard of perturbation method in potential theory is not justified. Müller-Kirsten et al. [9] and Sharma et al. [10-11] have presented a unified picture of a perturbative treatment for almost any type of nonsingular interactions-unified in the sense- that almost all the problems were solved with one and the same perturbation approach.

Large coupling solutions to various problems in potential theory and in particle physics have attracted considerable interest, particularly in view of the desire to understand strong interactions of purely field-theoretic formulations. However, attempts in this direction were severely handicapped due to the fact that the relevant coupling parameter is larger by a factor of about 1,000 than the fine structure constant of quantum electrodynamics. Since many expansions of physically relevant quantities currently studied are asymptotic in nature, there seems to be no plausible reason of ignoring asymptotic large coupling solutions. In fact, some time ago it has been emphasized by Dingle [12] that it is precisely the physicist's approach to a problem which will commonly lead to asymptotic rather than convergent expansions. Large coupling solutions of the Bethe-Salpeter equation of scalar ϕ^3 theory have been studied by Cheng and Wu [13]. Large coupling solutions of Bethe-Salpeter equation of the Wick-Cutkoksy model have been derived [14]. Kaushal [15], using high energy perturbation theory obtained eigen energy expansion for the interaction $\frac{\lambda x^2}{\left(1+gx^2\right)}$ in the range of small values of g and large values of λ. Sharma et al. [16] have recently formulated a perturbation technique for large coupling constant and used it to obtain the solutions of a double exponential potential in the Schrödinger equation. They have also shown a possibility that this formulation could be used for the solution of certain singular potentials. Bhargava and Sharma [17] have evaluated eigen energy expansions for a strong coupling harmonic potential superposed with weakly coupled linear and cubic anharmonicities.

In this paper, we evaluate expansions for eigen energies in terms of an odd integer quantum number

$q = 4n + 3 \; (n = 0,1,2,3......)$

for the potential

$$V(r) = -g^2 r^2 \left(1 + \lambda r^{2N}\right) \qquad (1)$$

Here g^2, the coupling constant is large, $N \geq 0$ and λ is small.

It is relevant here to point out that at times it is convenient to express the overall coupling constant in terms of a parameter β defined by

$$g = |\beta| e^{i\frac{\pi}{2}} \qquad (2)$$

We also discuss the applications of our general result in different cases and show in particular that the results for harmonic oscillator with positive quartic anharmonicities are in agreement with the results of Hioe and Montroll [18] (Table 1). In Fig. 1 a graph for the ground and a few of the excited energy levels of the quartic anharmonic oscillator, as a function of the anharmonicity parameter G, has been plotted. The behaviour of all the energy levels with respect to the anharmonicity parameter shows a qualitative similarity to the ground state of the oscillator. This observation is in agreement with that of Biswas et al. [19].

Table1. Ground state Energy Levels for Quartic Anharmonoc Oscillator for Different values of G.

G	$q=3$	$q=7$	$q=11$	$q=15$
0.0001	0.50007	1.50037	2.50097	3.50188
0.0005	0.50037	1.50186	2.50487	3.50938
0.0007	0.50053	1.50263	2.50683	3.51313
0.0009	0.50068	1.50338	2.50878	3.51688
0.001	0.50075	1.50375	2.50975	3.51875
0.0017	0.50128	1.50638	2.51658	3.53188
0.002	0.50150	1.50750	2.51950	3.53750
	(0.5014)	(1.5074)	(2.5192)	
0.003	0.50250	1.51120	2.52925	3..55625
0.004	0.50300	1.51500	2.53900	3.57500
0.005	0.50375	1.51875	2.54875	3.59375
0.006	0.50450	1.52200	2.5588	3.61250
	(0.5044)	(1.5218)	(2.5559)	
0.007	0.50525	1.52625	2.56825	3.63125
0.008	0.5060	1.53000	2.5780	3.65000
0.009	0.50675	1.53375	2.58775	3.66875
0.01	0.5075	1.53750	2.5975	3.68750
	(0.5072)	(1.5356)	(2.5908)	
0.02	0.51500	1.57500	2.69500	3.87500
0.03	0.52250	1.61250	2.79250	4.06250
0.04	0.53000	1.65000	2.89000	4.25000
0.05	0.5375	1.68750	2.9875	4.43750
	(0.5326)	(1.6534)	(2.8739)	
0.06	0.54500	1.72500	3.08500	4.62500
0.07	0.55250	1.76250	3.18250	4.81250
0.08	0.5600	1.80000	3.2800	5.00000
0.09	0.56750	1..83750	3.37750	5.18750
0.1	0.5750	1.87500	3.47500	5.37500
	(0.5591)	(1.76950)	(3.13860)	
0.2	0.65000	2.25000	4.45000	7.25000
0.3	0.72500	2.62500	5.42500	9.12500

It should be noted that the values given in the parentheses of Table I are from Reference [18].

2. THEORY

Consider the radial Schrödinger equation

$$\psi'' + \left[k^2 - \frac{l\,(l+1)}{r^2} - V(r)\right]\psi(r) = 0$$

$$\left(\hbar = c = 1,\ m = \frac{1}{2}\right) \qquad (3)$$

On substituting Eq. (1) in Eq. (3), we get

$$\frac{d^2\psi}{dz^2} + \left[\frac{k^2}{2ig} - \frac{l(l+1)}{z^2} - \frac{z^2}{4}\right]\psi(z)$$

$$= \frac{\lambda}{2}\left(\frac{1}{ig}\right)^N \left(\frac{z^2}{2}\right)^{N+1} \psi(z) \qquad (4)$$

where

$$z = (2ig)^{\frac{1}{2}}\ r$$
$$k^2 = E \qquad (5)$$

We wish to determine the approximate behavior of eigenenergy k^2 under normal bound state boundary conditions for large values of g. Here we use the perturbation procedure employed in our previous investigations [6-8,10-11] . In the limit $g \to \infty$, Eq. (4) may be approximated by

$$\frac{d^2\psi_q^{(0)}}{dz^2} + \left[\frac{k^2}{2ig} - \frac{l(l+1)}{z^2} - \frac{z^2}{4}\right]\psi_q^{(0)} = 0 \qquad (6)$$

where $\psi_q^{(0)}(z) = D_{1/2}(q-1)$ is a parabolic cylindrical function. The square integrability of $\psi_q^{(0)}$ demands that q be an odd integer, i.e., $q = 4n+3$, with $n = 0,1,2.......$ In Eq. (4) we now set

$$h = \frac{1}{ig} \text{ and } k^2 = ig(2l+q) - 2\Delta \qquad (7)$$

where Δ is yet to be determined and $h<1$. Substituting Eq. (7) in Eq. (4) we obtain

$$D_q\psi = \left\{\frac{\Delta h}{2} + \frac{\lambda}{4}h^{N+1}\left(\frac{z^2}{2}\right)^{N+1}\right\}\psi, \quad N > 1 \qquad (8)$$

with

$$D_q = \frac{1}{2}\left[\frac{d^2}{dz^2} + l + \frac{q}{2} - \frac{l(l+1)}{z^2} - \frac{z^2}{4}\right] \qquad (9)$$

which in the lowest order satisfies $D_q\psi_q^{(0)} = 0$. Now $\psi_q^{(0)}$ satisfies the recurrence relation

$$\left(\frac{z^2}{2}\right)\psi(a) = (a,a+1)\psi(a+1) + (a,a)\psi(a)$$
$$+ (a,a-1)\psi(a-1) \qquad (10)$$

where

$$\psi_q(z) = \psi(a,b,z) = \psi(a)$$

with

$$(a,a+1) = a = -\tfrac{1}{4}(q-3)$$
$$(a,a) = b - 2a = l + \tfrac{q}{2}$$
$$(a,a-1) = a - b = -\tfrac{1}{4}(q+3) - l \qquad (11)$$

In general we write,

$$\left(\frac{z^2}{2}\right)^m \psi(a) = \sum_{j=-m}^{m} S_m(a,a+j)\psi(a+j) \qquad (12)$$

where the coefficient $S_m(a,a+r)$ satisfies the following recurrence relation

$$S_m(a,a+r) = S_{m-1}(a,a+r)(a+r-1,a+r)$$
$$+ S_{m-1}(a,a+r)(a+r,a+r)$$
$$+ S_{m-1}(a,a+r+1)(a+r+1,a+r) \qquad (13)$$

The lowest order approximation leaves uncompensated the contribution

$$R_q^{(0)} = \frac{\Delta h}{2} + \frac{\lambda}{4}(h)^{N+1}\left(\frac{z^2}{2}\right)^{N+2}$$
$$= h^{N+1} \sum_{j=-(N+2)}^{N+2} [a, a+j]_{N+1} \quad \psi(a+j) \tag{14}$$

where

$$[a,a]_N = \frac{\Delta}{2} + \frac{\lambda}{4} S_{N+1}(a,a)$$

for $j = 0$ (15)

and

$$[a, a+j]_{N+1} = \frac{\lambda}{4} S_{N+2}(a, a+j)$$

for $j \neq 0$ (16)

Now the first order correction to the wave function is given by

$$\psi_q^{(1)} = h^{N+1} \sum_{\substack{j=-(N+2)\\ j\neq 0}}^{N+2} \frac{[a, a+j]_{N+1}}{j} \psi(a+j) \tag{17}$$

Again, in the first order, the uncompensated terms left are given by

$$R_q^{(1)} = h^{N+1} \sum_{\substack{j=-(N+2)\\ j\neq 0}}^{N+2} \frac{[a, a+j]_{N+1}}{j} h^{N+1} \times$$
$$[a+j, a]_{N+1} \psi(a+j) \tag{18}$$

The highest order corrections to the eigen functions, i.e. $\psi_q^{(2)}, \psi_q^{(3)}$ are obtained in a manner analogous to the derivation of $\psi_q^{(1)}$. Then adding the successive contributions, we obtain

$$\psi_q = \psi_q^{(0)} + \psi_q^{(1)} + \psi_q^{(2)} + \ldots\ldots \tag{19}$$

which can be written as

$$\psi_q = h^{N+1} \sum_{j=-(N+2)}^{N+2} C_N(a,j)\psi(a+j) \tag{20}$$

where the coefficients $C_N(a,j)$ follow by comparison. For Eq. (20) to be the solution of Eq. (8), the sum of the coefficients of $\psi(a)$ in $R_q^{(0)}, R_q^{(1)}$ ....left uncompensated so far must be set equal to zero, i.e. from Eq. (15) and Eq. (18) etc, for $N = 1$, we obtain

$$h^2 [a,a]_1 + h^4 \sum_{\substack{j=-3\\ j\neq 0}}^{3} \frac{[a, a+j]_2 [a+j, a]_2}{j} = 0 \tag{21}$$

This is the equation which determines Δ and hence eigen values E. The coefficients $[a, a+j]_{N+1}$ can now be evaluated with the help of Eqs. (11), (12), (13) and (16).

The values of E thus determined are

$$E = k^2 = ig(2l+q) + \frac{3\lambda}{8}(q^2+1)$$
$$-\frac{h^2\lambda^2}{2^{14}}\left[6288q^5 + 132640q^3 + 308432q\right]$$
$$+ O(\lambda^3) \tag{22}$$

In terms of $|\beta|$, Eq. (22) can be written as

$$k^2 = -\beta(2l+q)q + \frac{3\lambda}{8}(q^2+1)$$
$$+\frac{\lambda^2}{\beta^2}\frac{1}{2^{14}}\left[6288\,q^5 + 132640\,q^3 + 308432\,q\right]$$
$$+ O(\lambda^3) \tag{23}$$

Eqs. (20) and (22) which are the main results of this paper represent the asymptotic expansions for the large values of g and for small values of λ. The solution of Eq. (22) is valid for $h < 1$.

3. Applications

3.1 Harmonic Oscillator

The potential in Eq. (1) for $\lambda = 0$ takes the form

$$\begin{aligned} V(r) &= -g^2 r^2 \\ &= (ig)^2 r^2 \end{aligned} \tag{24}$$

Let

$$V(r) = \alpha^2 r^2$$

From Eq. (22)

$$k^2 = (2l + q) ig$$

Since

$$q = 4n + 3$$

$$l = \frac{k^2}{2ig} - 2n - \frac{3}{2} \tag{25}$$

From Eq. (25) ground state energy for the oscillator would be

$$\begin{aligned} k^2 &= \frac{3}{2}(2ig) \\ &= \frac{3}{2}(2\alpha) \end{aligned} \tag{26}$$

The result presented in Eq. (26) agrees with that of Sharma et al. [20].

3.2 Anharmonic Oscillator

The study of quantum mechanics of anharmonic oscillator is of considerable interest both from the physical as well as the mathematical point of view. The mathematical analysis of many of the oscillatory systems which exist in nature leads to the solution of non-linear differential equations. These solutions in turn have been quite helpful in understanding various problems encountered in quantum field theory and molecular physics. Precisely for this reason Sharma et al. [21] derived analytical formulas for the energy eigenvalues $E_n(\lambda)$ of one-dimensional anharmonic oscillators characterized by the potential $\omega^2 x^2 + \sum_{\alpha=2}^{m} \lambda_\alpha x^{2\alpha}$. These energy values, over a wide range of the values of n and λ, agreed well with the numerical values calculated by earlier workers. Bender and Wu [22] discussed anharmonic oscillator with a potential of the form

$$V(r) = \frac{x^2}{4} + \frac{Gx^4}{4} \tag{27}$$

In our case the equation of anharmonic oscillator potential is

$$V(r) = -g^2 r^2 - \lambda g^2 r^4 \tag{28}$$

Comparison of Eq. (27) with Eq. (28), yields

$$\lambda = G$$

and

$$ig = \frac{1}{2} \tag{29}$$

Substituting these values in Eq. (22), gives

$$k^2 = \left(2n + \frac{3}{2}\right) + 3G\left(2n^2 + 3n + \frac{5}{4}\right) + \mathrm{O}(G^2) \tag{30}$$

The results presented in Eq. (30) are also in agreement with those obtained by Bender and Wu [22]. Further for the one-dimensional case, Eq. (30) takes the form

$$\begin{aligned} E_n(G) &= \left(n + \frac{1}{2}\right) + \frac{3}{4} G(2n^2 + 2n + 1) + \mathrm{O}(G^2) \\ &= \left(n + \frac{1}{2}\right) + A_n(G) \end{aligned} \tag{31}$$

where

$$A_n(G) = \frac{3}{4} G \text{ for } n = 0$$

$$A_n(G) = \frac{15}{4} G \text{ for } n = 1 \tag{32}$$

In general,

$$A_n(G) = \frac{3G}{4}[1 + 2n(n+1)] + \mathrm{O}(G^2) \tag{33}$$

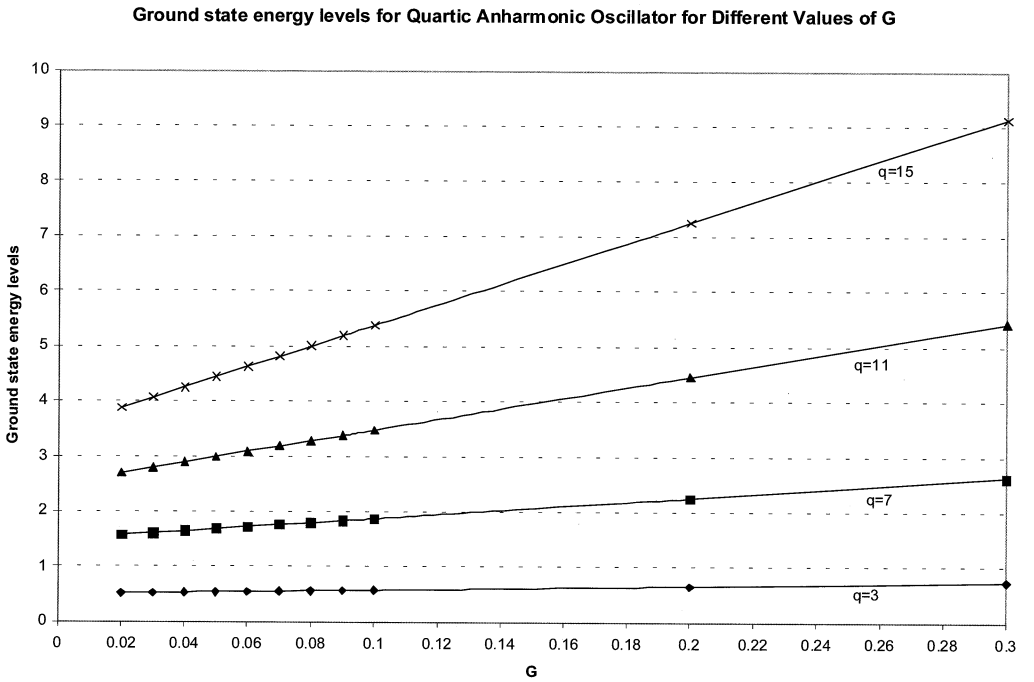

Figure 1. A graph for the ground and a few of the excited energy levels of the Quartic anharmonic oscillator as a function of the anharmonicity parameter G.

The above result is similar to the one derived by Hioe and Montroll [18].

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